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Linear Algebra

Basic Concepts of Matrix

Introduction

The word Matrix was introduced in 1860 by a French Mathematician Cayley. In Modern Mathematics, Engineering Mathematics has played an important role.

Definition

A set of mn numbers arranged in the form of a rectangular array where m is number of rows and n is number of columns, is called a matrix of $m \times n$ order. This is usually denoted by

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots\ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots\ldots a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & a_{m3} & \ldots\ldots a_{mn} \end{bmatrix}$$

The numbers $a_{11}, a_{12}, \ldots, a_{mn}$ are called the elements of the matrix. In the above concept of matrix, we can represent it by $A = [a_{ij}]$ where $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$ or simply by $A = [a_{ij}]_{m \times n}$. In the matrix, the vertical lines are called columns or column vectors and horizontal lines are called rows or row vectors. The element a_{ij} belongs to the ith row and the jth column, *i.e.,* the first suffix belongs to number of rows and second suffix belongs to number of columns. *e.g.*, if we write a_{43}, a_{43}, then it denotes the element of intersection of 4th row and 3rd column. Here, there are few examples of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}, \quad B = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3 \times 2}$$

and
$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix}_{3 \times 3}$$

Types of Matrices w.r.t. Size

(i) **Row Matrix**

A matrix of $1 \times n$ order is called row matrix, *i.e.,* a matrix which has only one row and any number of columns is called row matrix.

e.g., $[a \ \ b \ \ c \ \ d]_{1 \times 4}$ is a row matrix of order 1×4.

(ii) **Column Matrix**

A matrix of $m \times 1$ order is called column matrix or a matrix which has only one column and any number of rows is called column matrix.

e.g., $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{3 \times 1}$ is a column matrix of order 3×1.

(iii) Rectangular Matrix

A matrix is said to be rectangular matrix if the number of rows is different from the number of columns. There are two types of rectangular matrices :

(a) **Vertical matrix** If in a matrix, the number of rows is more than the number of columns, then it is called a vertical matrix. In other words, the matrix $[a_{ij}]_{m \times n}$ is a vertical matrix, if $m > n$.

e.g., $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3 \times 2}$ is a vertical matrix.

(b) **Horizontal matrix** If in a matrix, the number of rows is less than the number of columns, then it is called a horizontal matrix. In other words, the matrix $[a_{ij}]_{m \times n}$ is horizontal matrix, if $m < n$.

e.g., $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2 \times 3}$ is a horizontal matrix.

(iv) Square Matrix

A matrix is said to be square matrix, if the number of rows is equal to the number of columns. In other words, the matrix is a square matrix of order n.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

The elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called diagonal elements and the line going from left hand top corner to right hand bottom corner having elements $a_{11}, a_{22}, \ldots, a_{nn}$ is called principal diagonal of the square matrix.

Types of Matrices w.r.t. Virtue

(i) Null Matrix or Zero Matrix

Any matrix, in which all the elements are zero, is called zero matrix or null matrix. In other words, the matrix $A = [a_{ij}]_{m \times n}$ is null matrix, if $a_{ij} = 0, \forall\ i, j$.

e.g., $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3 \times 4}$ is zero matrix of order 3×4.

(ii) Diagonal Matrix

A square matrix whose all elements except the main diagonal elements are zero, is called diagonal matrix.

e.g., $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$ → Main diagonal

is a diagonal matrix.

Properties of Diagonal Matrix

(a) diag $[x, y, z]$ + diag $[p, q, r]$ = diag $[x + p, y + q, z + r]$

(b) diag $[x, y, z] \times$ diag $[p, q, r]$ = diag $[xp, yq, zr]$

(c) $\{\text{diag } [x, y, z]\}^n = \text{diag } [x^n, y^n, z^n]$

(d) $\{\text{diag } [x, y, z]\}^{-1} = \text{diag } \left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right]$

(e) $\{\text{diag } [x, y, z]\}^1 = \text{diag } [x, y, z]$

(f) Every zero matrix is a diagonal matrix.

(iii) Scalar Matrix

A diagonal matrix $A = [a_{ij}]$ is said to be a scalar matrix if all its diagonal elements are equal.

e.g., $\begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix}_{3 \times 3}$ is a scalar matrix.

Thus, every scalar matrix is a diagonal matrix.

> If $k = 1$, then scalar matrix is called identity matrix.

(iv) Unit or Identity Matrix

A diagonal matrix is said to be unit or identity matrix if its all diagonal elements are unit (1). It is represented by I_n.

e.g., $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4 \times 4}$$

are examples of identity matrices.

Properties of Identity Matrix

(a) I is identity element for multiplication so it is called multiplicative identity.

(b) $AI = IA = A$

(c) $I^n = I$

(d) $I^{-1} = I$

(e) $|I| = 1$

(f) Every identity matrix is a scalar matrix and hence diagonal matrix.

(v) Triangular Matrix

Triangular matrix is of two types as follows :

(a) **Upper triangular matrix** A square matrix is said to be upper triangular matrix, if all its elements below the main diagonal are zero.

e.g., $$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

is an upper triangular matrix.

(b) **Lower triangular matrix** A square matrix is said to be lower triangular matrix, if all its elements above the main diagonal are zero.

e.g., $$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 3 & 2 & 6 \end{bmatrix}$$

is a lower triangular matrix.

Algebra of Matrices

Addition of Matrices

Let P and Q are two matrices of same order $(m \times n)$. Then, their sum $(P + Q)$ is defined to be the matrix of the same order $(m \times n)$ obtained by adding the corresponding elements of given martices P and Q, *i.e.*,

If $P = [p_{ij}]_{m \times n}$ and $Q = [q_{ij}]_{m \times n}$

Then, $P + Q = [p_{ij} + q_{ij}]_{m \times n}$

In other words,

if $$P = \begin{bmatrix} p_{11} & p_{12} & \cdots\cdots & p_{1n} \\ p_{21} & p_{22} & \cdots\cdots & p_{2n} \\ \vdots & & & \\ p_{m1} & p_{m2} & \cdots\cdots & p_{mn} \end{bmatrix}_{m \times n}$$

and $$Q = \begin{bmatrix} q_{11} & q_{12} & \cdots\cdots & q_{1n} \\ q_{21} & q_{22} & \cdots\cdots & q_{2n} \\ \vdots & & & \\ q_{m1} & q_{m2} & \cdots\cdots & q_{mn} \end{bmatrix}_{m \times n}$$

Then,

$$P + Q = \begin{bmatrix} p_{11} + q_{11} & p_{12} + q_{12} & \cdots\cdots & p_{1n} + q_{1n} \\ p_{21} + q_{21} & p_{22} + q_{22} & \cdots\cdots & p_{2n} + q_{2n} \\ \vdots & \vdots & & \vdots \\ p_{m1} + q_{m1} & p_{m2} + q_{m2} & \cdots\cdots & p_{mn} + q_{mn} \end{bmatrix}_{m \times n}$$

Example 1. If $A = \begin{bmatrix} 1 & 4 & -3 \\ 0 & 8 & 9 \\ -4 & 2 & 0 \end{bmatrix}_{3 \times 3}$

and $B = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 0 & 3 \\ 1 & 2 & 5 \end{bmatrix}_{3 \times 3}$

Then, find $A + B$.

Sol. $\therefore$ $$A + B = \begin{bmatrix} 1+8 & 4+7 & -3+6 \\ 5+0 & 8+0 & 9+3 \\ -4+1 & 2+2 & 0+5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 11 & 3 \\ 5 & 8 & 12 \\ -3 & 4 & 5 \end{bmatrix}_{3 \times 3}$$

> Addition of two matrices is possible only when they are of same order.
> Similarly, the difference of two matrices is a matrix whose elements are obtained by subtracting the elements of 2nd matrix from the corresponding elements of 1st matrix.

Properties for Addition of Matrices

(a) **Commutative property** If P and Q are two matrices of same order $m \times n$, then

$$P + Q = Q + P.$$

(b) **Associative property** If P, Q and R be three matrices of same order $m \times n$, then

$$(P + Q) + R = P + (Q + R)$$

(c) **Additive identity** For each matrix P, there exists a null matrix 'O' of same order as P, such that

$$P = P + O = O + P$$

(d) **Additive inverse** Let P be a matrix of order $m \times n$, then the negative of the matrix P is called additive inverse of P and it is defined as $(-P)$.

(e) **Cancellation law** If P, Q and R be three comparable matrices of same order $m \times n$, then

If $P + Q = P + R \Rightarrow Q = R$ [Left cancellation law]

and $Q + P = R + P \Rightarrow Q = R$

[Right cancellation law]

Scalar Multiplication of Matrices

Let A be an $m \times n$ matrix and λ be a scalar, then λA is defined as $m \times n$ matrix whose each element is λ times. The corresponding elements of the matrix A, known as scalar multiplication of matrix A with λ.

e.g., if $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then

$$8A = \begin{bmatrix} 8 & 16 & 24 \\ 32 & 40 & 48 \end{bmatrix}$$

where, $\lambda = 8$

Properties of Scalar Multiplication

Let A and B be two matrices of same order and $\lambda, \lambda_1, \lambda_2$ are scalars. Then

(i) $\lambda (A \pm B) = \lambda A \pm \lambda B$

(ii) $(\lambda_1 + \lambda_2) A = \lambda_1 A + \lambda_2 A$

(iii) $\lambda_1 (\lambda_2 A) = (\lambda_1 \lambda_2) A$

(iv) $(-\lambda A) = -(\lambda A) = \lambda (-A)$

Multiplication of Matrices

The product of two matrices is possible only when the number of columns of 1st matrix is equal to the number of rows of 2nd matrix, where 1st matrix is called pre-factor and 2nd matrix is called post-factor and these types of matrices are called conformable for multiplication.

e.g., if
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3 \times 3}$$

and
$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3 \times 2}$$

$\because$ The matrix A is of order 3×3 and matrix B is of order 3×2.

i.e., Number of columns of A = Number of rows of B
Hence, the product AB is possible.

On the other hand, the number of columns of $B \neq$ number of rows of A, hence the product BA is not possible.

$$\text{Now, } AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} \end{bmatrix}_{3 \times 2}$$

Properties for Multiplication of Matrices

(a) **Commutative property** Multiplication of two matrices is not commutative, *i.e.,* if P and Q are two matrices, then

$$PQ \neq QP$$

Infact, if the product PQ exists, then it is not necessary that the product QP will also exist.

e.g.,
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 3 \\ 7 & 8 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

Then, AB is possible because the number of columns of A = number of rows of B

i.e.,
$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 3 \\ 7 & 8 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+9 & 5+6+12 \\ 4+0+9 & 20+0+12 \\ 7+16-3 & 35+24-4 \end{bmatrix} = \begin{bmatrix} 14 & 23 \\ 13 & 32 \\ 20 & 55 \end{bmatrix}$$

and BA is not possible because

Number of columns of $B \neq$ Number of rows of A

(b) **Associative property** Matrix multiplication is associative if conformability is assured.

i.e., $P(QR) = (PQ)R$ exists only when P, Q and R have $m \times n$, $n \times p$, $p \times q$ orders respectively.

(c) **Distributive law** Multiplication of matrices is distributive with respect to addition of matrices.

i.e., $P(Q + R) = PQ + PR$

and $(Q + R)P = QP + RP$

(d) **Cancellation law** If P, Q and R be three comparable matrices of same order $m \times n$, then if

$$AB = AC \Rightarrow B = C$$

It is possible only when A is non-singular matrix.

Equal Matrices

Two matrices A and B are said to be equal if they are of same order and corresponding elements are same *i.e.,* $a_{ij} = b_{ij}$.

Example 2. Find the values of a, b and c which satisfy

$$\begin{bmatrix} a+4 & 2a+c \\ b-3 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ 5 & 4 \end{bmatrix}$$

Sol. Since, the given matrices are equal, therefore their corresponding elements are also equal *i.e.,*

$$a+4=6 \Rightarrow a=2$$
$$b-3=5 \Rightarrow b=8$$
$$2a+c=9 \Rightarrow c=5$$

Trace of a Matrix

Let $A=[a_{ij}]_{m\times m}$ be the square matrix of order m. The sum of the elements lying along principal diagonal is called the trace of A and denoted by tr(A).

i.e., If $A=[a_{ij}]_{m\times m}$

Then, $\text{tr}(A)=\sum_{i=1}^{m} a_i, i=a_{11}+a_{22}+a_{33}+\ldots+a_{mm}$

e.g., Let $A=\begin{bmatrix}-1 & 5 & 6\\ 2 & 3 & -4\\ 8 & -1 & 9\end{bmatrix}$

Then, $\text{tr}(A)=-1+3+9=11$

Properties of Trace of Matrix

Let A and B be two square matrices of order m and λ be scalar. Then,

(i) $\text{tr}(\lambda A)=\lambda\,\text{tr}(A)$

(ii) $\text{tr}(A+B)=\text{tr}(A)+\text{tr}(B)$

(iii) $\text{tr}(A-B)=\text{tr}(A)-\text{tr}(B)$

(iv) $\text{tr}(AB)=\text{tr}(BA)$

Determinant of a Matrix

Let $A=[a_{ij}]_{n\times n}$ be any square matrix, then the determinant of A is denoted by det (A) or $|A|$ and defined as

$$A=\begin{vmatrix}a_{11} & a_{12} & a_{13} & \ldots\ldots a_{1n}\\ a_{21} & a_{22} & a_{23} & \ldots\ldots a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n1} & a_{n2} & a_{n3} & \ldots\ldots a_{nn}\end{vmatrix}_{n\times n}$$

The determinant has always a real finite value. If we define a 2×2 determinant, then it has two rows and two columns and its value is gives as follows

$$\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{21}a_{12}$$

The determinant higher than the order 2 are evaluated by using minors and cofactors.

Minors and Cofactors

Let A be any determinant of order n. If we eliminate the ith row and jth column of given determinant, then we obtain a determinant of order $(n-1)$. This determinant of $(n-1)$ order is called minor of given determinant. The cofactors of the element a_{ij} is defined as

$$C_{ij}=(-1)^{i+j}M_{ij}$$

where, $M_{ij}=\text{Minor}$

Properties of Determinants

(a) The value of a determinant does not change when rows and columns are interchanged.

(b) If any row (or column) of a matrix A is completely zero, then $|A|=0$

Also, if any two rows (or columns) of a matrix A are identical, then $|A|=0$

(c) If any two rows or two columns of a determinant are interchanged, the value of determinant is multiplied by (–1).

(d) If all elements of the one row (or one column) of a determinant are multiplied by same number k the value of determinant is k times the value of given determinant.

(e) If A be n-rowed square matrix and k be any scalar, then $|kA|=k^n|A|$

(f) In a determinant the sum of the products of ten elements of any row (or column) with the cofactors of corresponding elements of any row or (column) is equal to the determinant value.

(g) In determinant the sum of the products of the elements of any row (or column) with the cofactors of corresponding elements of other row or column is zero.

(h) If the elements of a row (or column) of a determinant are added m times the corresponding elements of another row (or column), the value of determinant thus obtained is equal to the value of original determinant.

(i) $|AB|=|A|\times|B|$ and based on this we can say the following :

$$|A^n|=(|A|)^n$$

Singular Matrix

A square matrix is said to be singular matrix if determinant of matrix A is zero *i.e.,* $|A|=0$.

e.g., $A=\begin{bmatrix}2 & 6\\ 3 & 9\end{bmatrix}$ is a singular matrix as $|A|=18-18=0$.

Non-singular Matrix

A square matrix is said to be non-singular matrix or regular matrix if determinant of matrix A is non-zero *i.e.,*

$|A| \neq 0$. *e.g.*, $A = \begin{bmatrix} 2 & 6 \\ 3 & 5 \end{bmatrix}$ is non-singular matrix as $|A| = 10 - 18 = -8 \neq 0$.

Related Matrices

(i) Submatrix

A matrix obtained from a given matrix by deleting some row or column or both is called a submatrix.

e.g., If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then submatrices of A are

$$A_1 = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \ldots$$

(ii) Transpose of a Matrix

If in a given matrix, we interchange row by column or column by row, then the new matrix so obtained is called transpose of the given matrix and is represented by A^T or A'.

e.g., If $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}_{3\times 4}$

Then, transpose of $A = A^T$ or $A' = \begin{bmatrix} a & e & i \\ b & f & j \\ c & g & k \\ d & h & l \end{bmatrix}_{4\times 3}$.

Properties of Transpose of a Matrix

If P', Q' are the transpose of P and Q, respectively, then

(a) $(P')' = P$

(b) $(P+Q)' = P' + Q'$

(c) $(kP)' = kP'$, k being any complex number.

(d) $(PQ)' = Q'P'$

Adjoint of a Matrix

Let $A = [a_{ij}]$ be a square matrix of order $n \times n$ and c_{ij} be the cofactor of a_{ij}. Then, the transpose of the matrix of cofactors is known as adjoint of a matrix and denoted by adj (A). Thus,

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$

Then, Adjoint of $A = \text{adj}(A) =$ Transpose of cofactor matrix C_{ij}

$$= \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

e.g., If $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 7 \\ 8 & 0 & 9 \end{bmatrix}$, then

$$A_{11} = (-1)^2 \begin{vmatrix} 4 & 7 \\ 0 & 9 \end{vmatrix} = 36 - 0 = 36$$

$$A_{12} = (-1)^3 \begin{vmatrix} 3 & 7 \\ 8 & 9 \end{vmatrix} = -(27 - 56) = 29$$

$$A_{13} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 8 & 0 \end{vmatrix} = 0 - 32 = -32$$

$$A_{21} = (-1)^3 \begin{vmatrix} 2 & 5 \\ 0 & 9 \end{vmatrix} = -(18 - 0) = -18$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & 5 \\ 8 & 9 \end{vmatrix} = 9 - 40 = -31$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 2 \\ 8 & 0 \end{vmatrix} = -(0 - 16) = 16$$

$$A_{31} = (-1)^4 \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} = 14 - 20 = -6$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} = -(7 - 15) = 8$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2$$

$\therefore$The cofactor matrix of A is

$$\begin{bmatrix} 36 & 29 & -32 \\ -18 & -31 & 16 \\ -6 & 8 & -2 \end{bmatrix}$$

$\therefore$Adjoint of A is

$$\begin{bmatrix} 36 & -18 & -6 \\ 29 & -31 & 8 \\ -32 & 16 & -2 \end{bmatrix}$$

Properties of Adjoint of Matrix

Let A be the square matrix of order n and I be the identity matrix of same order n. Then,

(i) $A \cdot (\text{adj } A) = |A|\, I = (\text{adj } A) \cdot A$

$\Rightarrow$ $A \cdot \text{adj}(A) = |A|\, I$

$$= \begin{bmatrix} |A| & 0 & 0 & \cdots\cdots & & 0 \\ 0 & |A| & 0 & \cdots\cdots & & 0 \\ 0 & 0 & |A| & 0 & \cdots\cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & & \cdots\cdots \\ 0 & 0 & \cdots\cdots & \cdots\cdots & 0 & |A| \end{bmatrix}$$

Taking determinant on both sides, we get

$$|A \cdot \text{adj } A| = |A|^n$$

$$\Rightarrow \quad |A| \cdot |\text{adj } A| = |A|^n \quad (\because \ |A \cdot B| = |A| \cdot |B|)$$

$$\Rightarrow \quad |\text{adj } A| = |A|^{n-1} \quad \text{provided } |A| \neq 0$$

Thus, we have

$$|\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1}$$

$$= |A|^{(n-1)^2}$$

In fact $\quad |\underbrace{\text{adj adj} \ldots \text{adj}}_{r \text{ times}} A| = |A|^{(n-1)^r}$

(ii) $\quad \text{adj}(\text{adj } A) = |A|^{n-2} \cdot A$

(iii) If $|A| \neq 0$, then

$$A \cdot \left(\frac{1}{|A|} \text{adj } A\right) = I = \left(\frac{1}{|A|} \text{adj } A\right) \cdot A$$

(iv) If k is any scalar number, then

$$\text{adj}(kA) = (\text{adj } A) \cdot k^{n-1}$$

(v) $\quad \text{adj } A' = (\text{adj } A)'$

Inverse of a Matrix

Let A be a square matrix, then there exists another matrix B such that $AB = I = BA$, where I is a unit matrix, then matrix B is called inverse of matrix A and denoted by A^{-1} and defined as

$$A^{-1} = \frac{\text{adj}(A)}{|A|}, \text{ provided } |A| \neq 0$$

e.g., If $\quad A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, then

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$$

$$= 1(-28+30) - 0(-21-0) - 1(-18-0)$$
$$= 2 - 0 + 18 = 20 \neq 0$$

Now,

$$A_{11} = (-1)^2 \begin{vmatrix} 4 & 5 \\ -6 & -7 \end{vmatrix} = -28 + 30 = 2$$

$$A_{12} = (-1)^3 \begin{vmatrix} 3 & 5 \\ 0 & -7 \end{vmatrix} = -(-21-0) = 21$$

$$A_{13} = (-1)^4 \begin{vmatrix} 3 & 4 \\ 0 & -6 \end{vmatrix} = -18 - 0 = -18$$

$$A_{21} = (-1)^3 \begin{vmatrix} 0 & -1 \\ -6 & -7 \end{vmatrix} = -(0-6) = 6$$

$$A_{22} = (-1)^4 \begin{vmatrix} 1 & -1 \\ 0 & -7 \end{vmatrix} = -7 - 0 = -7$$

$$A_{23} = (-1)^5 \begin{vmatrix} 1 & 0 \\ 0 & -6 \end{vmatrix} = -(-6-0) = 6$$

$$A_{31} = (-1)^4 \begin{vmatrix} 0 & -1 \\ 4 & 5 \end{vmatrix} = 0 + 4 = 4$$

$$A_{32} = (-1)^5 \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix} = -(5+3) = -8$$

$$A_{33} = (-1)^6 \begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} = 4 - 0 = 4$$

Thus, $\quad \text{adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$$= \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{\text{adj } A}{|A|}$$

$$= \frac{1}{20} \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & -\frac{7}{20} & -\frac{2}{5} \\ -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$$

Properties of Inverse Matrix

(a) Inverse of a square matrix A exists iff A is non-singular *i.e.*, $|A| \neq 0$.

(b) Inverse of a square matrix if it exists is unique.

(c) A and B are inverse of each other iff $AB = BA = I$, where A and B are non-singular square matrices of same order.

(d) $AA^{-1} = A^{-1}A = I$, where A is non-singular square matrix.

(e) If A and B are square matrices of same order, then AB is invertible iff both A and B are non-singular and

$$(AB)^{-1} = B^{-1}A^{-1}$$

In general,

$$(ABCD \ldots XYZ)^{-1} = Z^{-1}\,Y^{-1}\,X^{-1} \ldots D^{-1}\,C^{-1}\,B^{-1}\,A^{-1}$$

where, $A, B, C, D, \ldots, X, Y, Z$ are invertible square matrices of same order.

(f) $(A^{-1})^{-1} = A$, where, A is non-singular square matrices.

(g) $(A')^{-1} = (A^{-1})'$, where A is non-singular square matrix.

(h) Only for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$= \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Special Matrices

(i) Symmetric Matrix

A square matrix is said to be symmetric, if for all values of i and j, $a_{ij} = a_{ji}$, *i.e.,* if the transpose of a matrix is equal to the given matrix, then the given matrix is called symmetric matrix.

i.e., $\quad A' = A$

e.g., $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 6 \\ 3 & 6 & 7 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 6 \\ 3 & 6 & 7 \end{bmatrix}$$

$\Rightarrow \quad A' = A$

Properties of Symmetric Matrix

(a) In a symmetric matrix the elements above the principal diagonal and below the principal diagonal are same.

(b) Square null matrix, diagonal matrix, scalar matrix and identity matrix are symmetric matrices.

(c) Every symmetric matrix is necessarily be square matrix.

(d) Transpose of symmetric matrix needs not be symmetric matrix. *i.e.,* $(A')' = A \neq A'$.

(e) Symmetric matrix may be singular or non-singular according as its determinant is zero or non-zero, respectively.

(f) If A is symmetric matrix, then kA is also symmetric matrix for any scalar k.

(g) If A and B are symmetric matrices of same order. Then $\lambda_1 A \pm \lambda_2 B$ are also be symmetric matrices for scalars λ_1 and λ_2.

(h) If A and B are both symmetric matrices of same order. Then, AB is symmetric matrix iff $AB = BA$.

(i) If A be any square matrix. Then, AA' and $A'A$ both will be symmetric matrix.

(j) If A and B are symmetric matrices of same order. Then $AB + BA$ must be symmetric matrix.

i.e.,

$$\begin{aligned}(AB + BA)' &= (AB)' + (BA)' \\ &= B'A' + A'B' \\ &= BA + AB \\ &= AB + BA\end{aligned}$$

(k) If A is symmetric matrix, then all positive integral powers of A are symmetric matrices. *i.e.,*

$$(A^n)' = A^n$$

(l) If A is any square matrix, then $A + A'$ is always symmetric matrix. *i.e.,*

$$\begin{aligned}(A + A')' &= A' + (A')' \\ &= A' + A \\ &= A + A'\end{aligned}$$

(ii) Skew-symmetric Matrix

A square matrix is said to be skew-symmetric matrix, if for all values of i and j, $a_{ij} = -a_{ji}$, *i.e.,* if the transpose of a matrix is negatively equal to the given matrix, then the given matrix is called skew-symmetric matrix.

i.e., $\quad A' = -A$

e.g., If $$A = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

Then, $$A' = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 4 \\ -3 & -4 & 0 \end{bmatrix}$$

$$A' = -A$$

Properties of Skew-symmetric Matrix

(a) In a skew-symmetric matrix, the elements below the principal diagonal are negatively equal to the elements above the principal diagonal.

(b) All the principal diagonal elements of a skew-symmetric matrix are zero *i.e.,* $a_{ij} = 0$. Let $A = [a_{ij}]_{n \times n}$ be a skew-symmetric matrix. Then,

$$a_{ij} = -a_{ji}$$

Putting $j = i$, we get

$$a_{ii} = -a_{ii}$$

$$\Rightarrow \quad 2a_{ii} = 0$$

$$\Rightarrow \quad a_{ii} = 0$$

(c) Square null matrix is a skew-symmetric matrix but none of diagonal matrix, scalar matrix and identity matrix is skew-symmetric matrix.

(d) Every skew-symmetric matrix is necessarily be square matrix.

(e) If A is a skew-symmetric matrix, then kA is also skew-symmetric matrix for any scalar k.

(f) If A and B are skew-symmetric matrices of same order, then $\lambda_1 A \pm \lambda_2 B$ are also skew-symmetric matrix for scalars λ_1 and λ_2. *i.e.,*

$$(\lambda_1 A + \lambda_2 B)' = \lambda_1 A' + \lambda_2 B'$$
$$= -\lambda_1 A - \lambda_2 B$$
$$= -(\lambda_1 A + \lambda_2 B)$$

(g) If A and B are both skew-symmetric matrices of same order, then AB is skew-symmetric iff $AB + BA = 0$. *i.e.,*

$$(AB)' = B'A'$$
$$= (-B)(-A)$$
$$= BA = -AB$$

(h) If A and B are symmetric matrices of same order, then $AB - BA$ must be skew-symmetric matrix. *i.e.,*

$$(AB - BA)' = (AB)' - (BA)'$$
$$= B'A' - A'B'$$
$$= BA - AB$$
$$= -(AB - BA)$$

(i) If A is any skew-symmetric matrix, then all positive odd integral powers of A are skew-symmetric matrix.
i.e., $(A^n)' = (A')^n = (-A)^n = -A^n$ ($\because$ n is odd)

(j) If A is any square matrix, then $A - A'$ is always skew-symmetric. *i.e.,* $(A - A')' = A' - (A')'$ $= A' - A = -(A - A')$.

(k) Any square matrix A is uniquely expressed as the sum of a symmetric matrix and a skew-symmetric matrix. The symmetric part is $\frac{1}{2}(A + A')$ and skew-symmetric part is $\frac{1}{2}(A - A')$. Thus,

$$A = \underbrace{\frac{1}{2}(A + A')}_{\text{Symmetric matrix}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew-symmetric matrix}}$$

e.g., Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then

$$A' = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$\therefore$ Symmetric part

$$= \frac{1}{2}(A + A')$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$$

and skew-symmetric part

$$= \frac{1}{2}(A - A')$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Therefore,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \underset{\text{Symmetric matrix}}{\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}} + \underset{\text{Skew-symmetric matrix}}{\begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}}$$

(iii) Orthogonal Matrix

A square matrix is said to be orthogonal matrix if the product of the matrix and its transpose is an identity matrix.

i.e., $AA' = A'A = I$

e.g., $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

are orthogonal matrices.

Properties of an Orthogonal Matrix

(a) If A is orthogonal matrix, then A is non-singular and $A^{-1} = A'$. *i.e.,* $A'A = I \Rightarrow A(A'A) = AI = A$ $\Rightarrow (AA')A = A \Rightarrow AA' = I \Rightarrow A$ is non-singular and $A^{-1} = A'$.

(b) If A is an orthogonal matrix, then $|A| = \pm 1$. *i.e.,* $A'A = I \Rightarrow |A'A| = 1 \Rightarrow |A'||A| = 1 \Rightarrow |A|^2 = 1$ $\Rightarrow |A| = \pm 1$.

(c) Transpose of an orthogonal matrix is orthogonal. *i.e.,* If A is orthogonal matrix, then A' is also orthogonal matrix. *i.e.,* $AA' = A'A = I \Rightarrow$ $(AA')' = (A'A)' = I \Rightarrow AA' = A'A = I$.

(d) Product of two orthogonal matrices is orthogonal matrix. *i.e.,* If A and B are orthogonal matrices, then AB and BA are orthogonal matrices. *i.e.,*

$$\begin{aligned}(AB)(AB)' &= (AB)(B'A') \\ &= A(BB')A' \\ &= AIA' \\ &= AA' = I\end{aligned}$$

Similarly,

$$\begin{aligned}(BA)(BA)' &= (BA)(A'B') \\ &= B(AA')B' \\ &= BIB' \\ &= BB' = I\end{aligned}$$

(e) Inverse of an orthogonal matrix is orthogonal. *i.e.,* if A is orthogonal matrix, then A^{-1} is also orthogonal. *i.e.,*

$$AA' = A'A = I$$

$$\Rightarrow \quad (AA')^{-1} = (A'A)^{-1} = I$$

$$\Rightarrow \quad (A')^{-1}A^{-1} = A^{-1}(A')^{-1} = I$$

$$\Rightarrow \quad (A^{-1})'A^{-1} = A^{-1}(A^{-1})' = I$$

Example 3. If $A = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$, then show that A is orthogonal.

Sol. $\because A' = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

From $AA' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta\cos\theta + \sin\theta\cos\theta & 0 & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \qquad [\because \sin^2\theta + \cos^2\theta = 1]$$

Hence, A is orthogonal matrix.

(iv) Conjugate of a Matrix

If $z = x + iy$ be a complex number where x and y are real numbers, then $\bar{z} = x - iy$ is called the conjugate of the complex number z.

The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex number is called the conjugate of A denoted by $\bar{A}$. Thus, if $A = [a_{ij}]_{m\times n}$, then $\bar{A} = [\bar{a}_{ij}]_{m\times n}$

e.g., If $A = \begin{bmatrix} 1+2i & 4 \\ 4-5i & -7i \end{bmatrix}$

Then, conjugate of $A = \bar{A} = \begin{bmatrix} 1-2i & 4 \\ 4+5i & 7i \end{bmatrix}$

Properties of Conjugate of a Matrix

Let $\bar{A}$ and $\bar{B}$ be the conjugate matrices of matrices A and B, respectively. Then,

(a) $\overline{\overline{A}} = A$

(b) $\overline{(A+B)} = \overline{A} + \overline{B}$

(c) $\overline{(kA)} = \bar{k}\ \overline{A}$, k being any scalar number.

(d) $\overline{AB} = \overline{A}\ \overline{B}$, A and B are conformable to multiplication.

(e) $\overline{(A^n)} = (\overline{A})^n$.

(f) $\overline{A} = A$ iff A is purely real matrix.

(g) $\overline{A} = -A$ iff A is purely imaginary matrix.

(v) Tranjugate or Transposed Conjugate of a Matrix

The transpose conjugate of a given matrix is obtained by interchanging the rows and columns of the matrix obtained by replacing the elements of A by their corresponding complex conjugate *i.e.,* the transpose conjugate of matrix A is $(\overline{A})'$ or $(\overline{A'})$ and denoted by A^θ. Thus, $A^\theta = (\overline{A})' = (\overline{A'})$.

e.g., Let $$A = \begin{bmatrix} 1+2i & 4 & i \\ 1-i & 2+i & 3-2i \end{bmatrix}$$

$$\Rightarrow \quad \overline{A} = \begin{bmatrix} 1-2i & 4 & -i \\ 1+i & 2-i & 3+2i \end{bmatrix}$$

$$\therefore \quad A^\theta = (\overline{A})' = \begin{bmatrix} 1-2i & 1+i \\ 4 & 2-i \\ -i & 3+2i \end{bmatrix}$$

Properties of Transpose Conjugate of Matrix

Let A^θ and B^θ are transpose conjugates of matrices A and B respectively. Then

(a) $(A^\theta)^\theta = A$

(b) $(A+B)^\theta = A^\theta + B^\theta$

(c) $(kA)^\theta = \bar{k} A^\theta$

(d) $(AB)^\theta = B^\theta A^\theta$, A and B are conformable for multiplication.

(e) $(A^n)^\theta = (A^\theta)^n$

(vi) Unitary Matrix

A square matrix A is said to be unitary matrix, if

$$AA^\theta = A^\theta A = I$$

e.g., Let $A = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$, then

$$A^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\therefore \quad AA^\theta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad AA^\theta = I$$

$\Rightarrow$ A is unitary matrix.

Properties of Unitary Matrix

(a) If A is an unitary matrix, then A' is also unitary matrix. *i.e.,*

$$AA^\theta = A^\theta A = I$$

$$\Rightarrow \quad (AA^\theta)' = (A^\theta A)' = I$$

$$\Rightarrow \quad (A^\theta)' A' = A' (A^\theta)' = I$$

$$\Rightarrow \quad (A')^\theta A' = A' (A')^\theta = I$$

(b) If A and B are two unitary matrices of same order, then AB and BA are also unitary matrices of same order. *i.e.,*

$$(AB)(AB)^\theta = AB\,(B^\theta A^\theta)$$
$$= A\,(BB^\theta)\,A^\theta$$
$$= AIA^\theta$$
$$= AA^\theta = I$$

Similarly,

$$(BA)(BA)^\theta = (BA)(A^\theta B^\theta)$$
$$= B\,(AA^\theta)\,B^\theta$$
$$= BIB^\theta$$
$$= BB^\theta = I$$

(c) If A is unitary matrix, then A^{-1} is also unitary matrix. *i.e.,*

$$AA^\theta = A^\theta A = I$$

$$\Rightarrow \quad (AA^\theta)^{-1} = (A^\theta A)^{-1} = I$$

$$\Rightarrow \quad (A^\theta)^{-1} A^{-1} = A^{-1} (A^\theta)^{-1} = I$$

$$\Rightarrow \quad (A^{-1})^\theta A^{-1} = A^{-1} (A^{-1})^\theta = I$$

(vii) Hermitian Matrix

If the conjugate transpose (A^θ) of a matrix is equal to the given square matrix, then this matrix is called Hermitian matrix, *i.e.,*

$$A^\theta = A$$

e.g., $$A = \begin{bmatrix} i & 2-3i & 3-i \\ 2+3i & 2 & 1+2i \\ 3+i & 1-2i & 5 \end{bmatrix}$$

Properties of Hermitian Matrix

(a) A square matrix $A = [a_{ij}]_{n\times n}$ is Hermitian iff $\bar{a}_{ij} = a_{ji}$.

(b) In a Hermitian matrix, the elements on the principal diagonal all are real numbers *i.e.,* $\bar{a}_{ii} = a_{ii}$. *i.e.,*

$$\bar{a}_{ij} = a_{ji}$$

Putting $j = i$, we get

$$\bar{a}_{ii} = a_{ii}$$

(c) If A is Hermitian matrix, then kA is also Hermitian matrix, k is any real number. *i.e.,* $(kA)^\theta = \bar{k}A^\theta = kA$.

(d) If A and B are Hermitian matrices of same order, then $\lambda_1 A \pm \lambda_2 B$ are also Hermitian matrices for any real scalars λ_1 and λ_2.

(e) If A and B are Hermitian matrices of same order, then AB is also Hermitian iff $AB = BA$. *i.e.,* $(AB)^\theta = B^\theta A^\theta = BA = AB$.

(f) If A be any square matrix, then AA^θ and $A^\theta A$ are also Hermitian matrices. *i.e.,* $(AA^\theta)^\theta = (A^\theta)^\theta A^\theta$ $= AA^\theta$.

Similarly, $(A^\theta A)^\theta = A^\theta (A^\theta)^\theta = A^\theta A$.

(g) If A and B are Hermitian matrices of same order, then $AB + BA$ is also Hermitian matrix. *i.e.,* $(AB + BA)^\theta = (AB)^\theta + (BA)^\theta = B^\theta A^\theta + A^\theta B^\theta$ $= BA + AB = AB + BA$.

(h) If A is a Hermitian matrix, then $\bar{A}$ is also Hermitian matrix.

(i) If A is any square matrix, then all positive integral powers of A are Hermitian matrix.

(j) If A is any square matrix, then $A + A^\theta$ is a Hermitian matrix.

(k) Any square matrix can be expressed as $A + iB$, where, A and B are Hermitian matrices.

(viii) Skew-Hermitian Matrix

A square matrix is said to be Skew-Hermitian matrix, if the conjugate transpose of a matrix is negatively equal to the given matrix.

i.e., $$A^\theta = -A$$

Properties of Skew-Hermitian Matrix

(a) A square matrix $A = [a_{ij}]_{n\times n}$ is Skew-Hermitian iff $\bar{a}_{ij} = -a_{ji}$.

(b) In a Skew-Hermitian matrix, the elements on the principal diagonal must be purely imaginary numbers are zero. *i.e.,*

$$\bar{a}_{ij} = -a_{ji}$$

Putting $j = i$, we get

$$\bar{a}_{ji} = -a_{ji}$$

$\Rightarrow$ $\bar{a}_{ii} + a_{ii} = 0$

$\Rightarrow$ Real part of $a_{ii} = 0$.

(c) If A is Skew-Hermitian matrix, then (kA) is also Skew-Hermitian matrix for any real scalar k. *i.e.,* $(kA)^\theta = \bar{k}A^\theta = -kA$.

(d) If A and B are Skew-Hermitian matrices of same order, then $\lambda_1 A \pm \lambda_2 B$ are also Skew-Hermitian matrices of same order for any real numbers λ_1 and λ_2.

(e) If A and B are Hermitian matrices of same order, then $AB - BA$ is Skew-Hermitian. *i.e.,* $(AB - BA)^\theta = (AB)^\theta - (BA)^\theta = B^\theta A^\theta - A^\theta B^\theta$ $= BA - AB = -(AB - BA)$.

(f) If A is any square matrix, then $A - A^\theta$ is Skew-Hermitian. *i.e.,* $(A - A^\theta)^\theta = A^\theta - (A^\theta)^\theta$ $= A^\theta - A = -(A - A^\theta)$.

(g) If A is a Skew-Hermitian matrix, then $\bar{A}$ is also Skew-Hermitian matrix.

(h) If A is Hermitian matrix, then iA is Skew-Hermitian matrix. *i.e.,* $(iA)^\theta = \bar{i}A^\theta = -iA$.

(i) If A is Skew-Hermitian matrix, then iA is Hermitian matrix. *i.e.,*

$$(iA)^\theta = \bar{i}A^\theta = (-i)(-A) = iA.$$

(j) Every square matrix can be uniquely represented as the sum of a Hermitian matrix and a Skew-Hermitian matrix *i.e.,*

$$A = \underbrace{\frac{1}{2}(A + A^\theta)}_{\text{Hermitian matrix}} + \underbrace{\frac{1}{2}(A - A^\theta)}_{\text{Skew-Hermitian matrix}}$$

(ix) Idempotent Matrix

A square matrix is said to be idempotent, if $A^2 = A$

Properties of Idempotent Matrix

(a) An idempotent matrix is necessarily be square matrix.

(b) If A and B are idempotent matrices of same order, then AB is idempotent iff $AB = BA$. *i.e.*, $(AB)^2 = (AB)(AB) = A(BA)B$

$$= A(AB)B = A^2B^2 = AB$$

(c) If A and B are idempotent matrices of same order, then $A + B$ is idempotent iff $AB = BA = O$. *i.e.*, $(A+B)^2 = A^2 + AB + BA + B^2$

$$= A^2 + B^2 = A + B. \quad (\because A^2 = A, B^2 = B)$$

(d) If A is an idempotent matrix and $A + B = I$, then B is idempotent and $AB = BA = O$. *i.e.*, $B = I - A$

$$\Rightarrow \quad B^2 = (I - A)^2 = I - A - A + A^2$$
$$= I - A$$
$$\Rightarrow \quad B^2 = I - A \Rightarrow B^2 = B$$

and $\quad AB = A(I - A) = A - A^2 = A - A = 0.$

Similarly, $BA = (I - A)A = A - A^2 = A - A = 0$

(e) Diag (1, 1, 1, ……, 1) is an idempotent matrix. *i.e.*, $[\text{Diag}(1, 1, 1, \ldots\ldots, 1)]^2 = \text{Diag}(1, 1, 1, \ldots\ldots, 1)$

(f) If $\lambda_1, \lambda_2, \lambda_3$ are direction cosines, then the matrix

$$\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_1\lambda_2 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_1\lambda_3 & \lambda_2\lambda_3 & \lambda_3^2 \end{bmatrix}$$

is an idempotent matrix.

(g) If $AB = A$ and $BA = B$, then A and B are idempotent matrices. *i.e.*, $AB = A \Leftrightarrow BAB = BA$ $\Leftrightarrow BAB = B \Leftrightarrow AB = I \Leftrightarrow A^{-1} = B$.

Thus, $AB = A \Leftrightarrow AA^{-1} = A \Leftrightarrow A = I$

$$\Leftrightarrow A^2 = I \Leftrightarrow A^2 = A.$$

Similarly, $BA = B \Leftrightarrow B = I$

$$\Leftrightarrow B^2 = I \Leftrightarrow B^2 = B$$

Example 4. Show that $\begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent matrix.

Sol. $\because$

$$A = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$\Rightarrow \quad A^2 = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, A is idempotent matrix.

(x) Involutory Matrix

A square matrix is said to be an involutory matrix, if $A^2 = I$.

Properties of Involutory Matrix

(a) Identity matrix is always involutory matrix.

(b) A is involutory matrix iff

$$(A - I)(A + I) = 0$$

i.e., $\quad A^2 = I \Leftrightarrow A^2 - I = 0$

$$\Leftrightarrow \quad (A - I)(A + I) = 0$$

Example 5. Show that $\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$ is involutory matrix.

Sol. From $A^2 = A \times A$,

$$= \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}\begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 16-3-12 & 12+0-12 & 12-3-9 \\ -4+0+4 & -3+0+4 & -3+0+3 \\ -16+4+12 & -12+0+12 & -12+4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A is involutory matrix.

(xi) Periodic Matrix

A matrix is said to be periodic, if $A^{n+1} = A$, where n is positive integer.

- If $k = 1$, then periodic matrix is called idempotent matrix.

(xii) Nilpotent Matrix

A matrix is said to be nilpotent, if $A^k = 0$, where k is any positive number.

Example 6. Prove that $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent of class 3.

Sol. From $A^2 = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1+5-6 & 1+2-3 & 3+6-9 \\ 5+10-12 & 5+4-6 & 15+12-18 \\ -2-5+6 & -2-2+3 & -6-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Now, $A^3 = A^2 - A$

$$\begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+0 & 0+0+0 & 0+0+0 \\ 3+15-18 & 3+6-9 & 9+18-27 \\ -1-5+6 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, A is nilpotent matrix of order 3.

Example 7. Show that any square matrix can be expressed as the sum of symmetric and anti-symmetric (skew-symmetric) matrix.

Sol. Let A be any given matrix.

Then, $2A = A + A$

or $2A = (A + A') + (A - A')$

$\Rightarrow$ $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Now, $(A + A')' = A' + A = A + A'$

$\Rightarrow (A + A')$ is a symmetric matrix.

Again from $(A - A')' = A' - A = -(A - A')$

$\Rightarrow (A - A')$ is skew-symmetric matrix.

Hence, $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

Matrix = Symmetric matrix + Skew-symmetric matrix

Example 8. Prove that the matrix

$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Sol. $\because$ $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$\overline{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1-i \\ 1+i & -1 \end{bmatrix}$$

$$A^{\theta} = (\overline{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

Now, $A^{\theta} \cdot A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, the given matrix 'A' is unitary matrix.

Example 9. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$, then find $3A - 4B$.

Sol. We have, $3A = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & 3 & 15 \end{bmatrix} \quad \ldots\text{(i)}$$

and $4B = 4\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{bmatrix} \quad \ldots\text{(ii)}$$

Now, subtracting Eq. (ii) from Eq. (i)

$$3A - 4B = \begin{bmatrix} 6 & 9 & 3 \\ 0 & 3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 27 \\ 0 & 7 & 3 \end{bmatrix}$$

Example 10. Show that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$ is orthogonal.

Sol. For orthogonality of a matrix 'A', we have to show that $A'A = AA' = I$

Hence,

$$A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

Now,

$$A'A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0. & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Hence, the given matrix 'A' is orthogonal matrix.

Example 11. Prove that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is idempotent.

Sol. For an idempotent matrix of a given matrix 'A', we have to show that

$$A^2 = A$$

So, $A^2 = A \times A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 4+2-4 & -4-6+8 & -8-8+12 \\ -2-3+4 & 2+9-8 & 4+12-12 \\ 2+2-3 & -2-6+6 & -4-8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, given matrix is an idempotent matrix.

Example 12. Express $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix}$ as the sum of a symmetric and skew-symmetric matrices.

Sol. $A' = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & -4 \\ 3 & -3 & 4 \end{bmatrix}$

Now, $A + A' = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & -4 \\ 3 & -3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & -7 \\ 1 & -7 & 8 \end{bmatrix} \quad \text{...(i)}$$

and $A - A' = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -2 \\ 1 & 3 & -4 \\ 3 & -3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 1 \\ -5 & -1 & 0 \end{bmatrix} \quad \text{...(ii)}$$

On adding Eqs. (i) and (ii), we get

$$2A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & -7 \\ 1 & -7 & 8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 1 \\ -5 & -1 & 0 \end{bmatrix}$$

or $A = \frac{1}{2}\begin{bmatrix} 2 & 2 & 1 \\ 2 & 6 & -7 \\ 1 & -7 & 8 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 1 \\ -5 & -1 & 0 \end{bmatrix}$

[Symmetric matrix] [Skew-symmetric matrix]

Example 13. Real matrices $[A]_{3\times1}, [B]_{3\times3}, [C]_{3\times5}, [D]_{5\times3}, [E]_{5\times5}$ and $[F]_{5\times1}$ are given.
Matrices $[B]$ and $[E]$ are symmetric.
Following statements are made with respect to these matrices :
(i) Matrix product $[F]^T [C]^T [B][C][F]$ is a scalar.
(ii) Matrix product $[D]^T [F][D]$ is always symmetric.
With reference to above statements, which of the following applies? **[CE GATE 2004]**
(a) Statement (i) is true but (ii) is false.
(b) Statement (i) is false but (ii) is true.
(c) Both the statements are true.
(d) Both the statements are false.

Sol. (d) Both the statements are false. (i) is false because the product of two or matrices is always a matrix and statement $[D]^T[F][D]$ does not exist because it is not compitable for matrix product. Hence, the statement (ii) is also false.

Example 14. Let $f(x) = x^2 - 5x + 6$, and $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then $f(A)$ is equal to

(a) $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -1 & -5 \\ -1 & -1 & 4 \\ -3 & -10 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 4 \\ -1 & 4 & -10 \\ 4 & -3 & -5 \end{bmatrix}$ (d) None of these

Sol. (a) $\because f(A) = A^2 - 5A + 6I$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

Example 15. If $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then A^2 is

(a) idempotent (b) nilpotent
(c) involutory (d) periodic

Sol. (c) $\because \quad A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, A^2 is involutory.

Example 16. The inverse of the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ is

[CE GATE 2007]

(a) $\frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$ (b) $\frac{1}{3}\begin{bmatrix} 7 & 2 \\ 5 & 1 \end{bmatrix}$

(c) $\frac{1}{3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$ (d) $\frac{1}{3}\begin{bmatrix} -7 & -2 \\ -5 & -1 \end{bmatrix}$

Sol. (a) Inverse of $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1}$

$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

So, inverse of $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix} = \frac{1}{-3}\begin{bmatrix} 7 & -2 \\ -5 & 1 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$$

Example 17. The product of matrices $(PQ)^{-1} P$ is

[CE GATE 2008]

(a) P^{-1} (b) Q^{-1}

(c) $P^{-1}Q^{-1}P$ (d) $PQ\,P^{-1}$

Sol. (b) $\because \quad (PQ)^{-1} P = (Q^{-1} \cdot P^{-1})P$

$$= Q^{-1}\,(P^{-1}P)$$

$$= Q^{-1}(I)$$

$$= Q^{-1}$$

Example 18. If $R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$, then top row of R^{-1} is

[CE GATE 2005]

(a) $[5\ \ 6\ \ 4]$ (b) $[5\ \ -3\ \ 1]$

(c) $[2\ \ 0\ \ -1]$ (d) $\left[2\ \ -1\ \ \frac{1}{2}\right]$

Sol. (b) $\because \quad R = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$

Then, $|R| = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{vmatrix}$

$$= 1\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$\because$ We have to find only the top row of R^{-1}, then we only find cofactors of the first column of given matrix, R.

So, cofactor of $a_{11} = +\begin{vmatrix} 1 & -1 \\ 3 & 2 \end{vmatrix} = 5$

Cofactor of $a_{21} = -\begin{vmatrix} 0 & -1 \\ 3 & 2 \end{vmatrix} = -(+3) = -3$

Cofactor of $a_{31} = +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} = -(-1) = +1$

So, matrix of Cofactors of $R = \begin{bmatrix} 5 & - & - \\ -3 & - & - \\ 1 & - & - \end{bmatrix}$

So, adjoint of $R = \text{adj}\,(R) = [\text{Cofactor } A]'$

$$= \begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

So, $R^{-1} = \dfrac{\text{adj}\,(R)}{|R|} = \dfrac{1}{1}\begin{bmatrix} 5 & -3 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$

Hence, upper row of $R^{-1} = [5\ \ -3\ \ 1]$.

Example 19. Multiplication of matrices E and F is G. Matrices E and G are as follows :

$$E = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then the value of matrix F is **[ME GATE 2006]**

(a) $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \cos\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \sin\theta & -\cos\theta & 0 \\ \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol. (c) $\because$ Given that $E \times F = G$

$$\Rightarrow \quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad F = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

[Because the product of any matrix and unit matrix is the matrix itself.]

So, $|E| = \cos\theta \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} - (-\sin\theta) \begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} \sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix}$

$= \cos^2\theta + \sin^2\theta = 1$

Matrix of Cofactors of $E = C = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So, adjoint of $E = C' = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Hence, $F = \dfrac{\text{adj}(E)}{|E|} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example 20. For which value of x will the matrix given below become singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

[ME GATE 2004]

(a) 4 (b) 6
(c) 8 (d) 12

Sol. (a) $\because$ A matrix is said to be singular, if $|A| = 0$

$\Rightarrow \begin{vmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{vmatrix} = 0$

$\Rightarrow 8(0-12) - x(0-24) + 0(24-0) = 0$

$\Rightarrow x = 4$

Example 21. Let $A = \begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix}$, then $a + b$ is equal to **[EC GATE 2005]**

(a) $\dfrac{7}{20}$ (b) $\dfrac{3}{20}$ (c) $\dfrac{19}{60}$ (d) $\dfrac{11}{20}$

Sol. (a) $\because$ For a square matrix

$AA^{-1} = I$

$\begin{bmatrix} 2 & -0.1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1/2 & a \\ 0 & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2a - 0.1b \\ 0 & 3b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow 2a - 0.1b = 0$ ...(i)

$3b = 1 \Rightarrow b = \dfrac{1}{3}$

Putting b in Eq. (i)

$2a - 0.1 \times \dfrac{1}{3} = 0$

$\Rightarrow a = \dfrac{1}{60}$

Hence, $a + b = \dfrac{1}{60} + \dfrac{1}{3} = \dfrac{7}{20}$

Intro Exercise 1

1. $(AB)^{-1}$ is equal to

(a) $A^{-1}B^{-1}$ (b) $B^{-1}A^{-1}$
(c) $(A')^{-1}(B')^{-1}$ (d) $A^{-1}(B')^{-1}$

2. Inverse of $\begin{bmatrix} 4 & 3 \\ -7 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} \frac{1}{4} & \frac{1}{3} \\ -\frac{1}{7} & 1 \end{bmatrix}$ (b) $\dfrac{1}{25}\begin{bmatrix} 4 & 3 \\ -7 & 1 \end{bmatrix}$

(c) $\dfrac{1}{25}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\dfrac{1}{25}\begin{bmatrix} 1 & -3 \\ 7 & 4 \end{bmatrix}$

3. In the matrix equation $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$, the values of x and y are

(a) $x = 3, y = -1$ (b) $x = 2, y = 5$
(c) $x = 1, y = -1$ (d) $x = -1, y = 1$

4. The characteristics of an orthogonal matrix A is

(a) $AA^{-1} = I$ (b) $AA^{\theta} = I$
(c) $AA' = I$ (d) $A^1 \cdot A^{-1} = I$

5. For any two matrices A and B, which is true?

(a) $AB = 0 \Rightarrow A = 0$ or $B = 0$
(b) $(A + B)^2 = A^2 + 2AB + B^2$
(c) $(AB)' = B'A'$
(d) $|AB| \neq |A||B|$

6. If $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 5 \\ 2 & -5 & 0 \end{bmatrix}$, then which is correct?

(a) Symmetric (b) Skew-symmetric
(c) Hermitian (d) Non-singular

7. Matrix $A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$ is

(a) idempotent (b) nilpotent
(c) involutory (d) periodic

8. The matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$ is

(a) singular (b) invertible
(c) symmetric (d) skew-symmetric

9. Matrix $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$ is

(a) unitary (b) orthogonal
(c) nilpotent (d) None of these

10. If A and B are symmetric matrices, then which is correct for $A+B$?

(a) Symmetric (b) Skew-symmetric
(c) Singular (d) Identity

11. A square matrix B is skew-symmetric, if

[CE GATE 2009]

(a) $B' = -B$ (b) $B' = B$
(c) $B^{-1} = B$ (d) $B^{-1} = B'$

12. Match List I with List II and select the correct answer using the codes given below in the lists.

List I	List II
A. Singular matrix	1. Determinant is not defined
B. Non-singular matrix	2. Determinant is always one
C. Real symmetric	3. Determinant is zero
D. Orthogonal matrix	4. Eigen values are always real
	5. Eigen values are not defined

[ME GATE 2006]

Codes

	A	B	C	D
(a)	3	1	4	2
(b)	2	3	4	1
(c)	3	2	5	4
(d)	3	4	2	1

13. Inverse of a symmetric matrix is

(a) symmetric (b) skew-symmetric
(c) orthogonal (d) None of these

14. If A is a square matrix,then which is not symmetric from the following?

(a) $A - A'$ (b) $A + A'$
(c) AA' (d) AA^{-1}

15. The inverse of a diagonal matrix is

(a) symmetric (b) skew-symmetric
(c) orthogonal (d) diagonal

16. If A and B are square matrices of same order such that $AB = A$ and $BA = B$, then A and B are

(a) singular (b) non-singular
(c) unit matrix (d) None of these

17. Given an orthogonal matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \text{ then } (AA')^{-1} \text{ is}$$

[EC GATE 2005]

(a) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$

18. If $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, then the value of A^4 is

(a) $\begin{bmatrix} 1 & a^4 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 4a \\ 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & a^4 \\ 0 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$

19. If the matrix $\begin{bmatrix} 1 & b & 2 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ is not invertible, then the value of b is

(a) 2 (b) 0 (c) 1 (d) –1

20. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to

(a) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (b) $\frac{1}{27}\begin{bmatrix} -1 & 26 \\ 0 & 27 \end{bmatrix}$

(c) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (d) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

21. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then A^{-1} is

(a) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) None of these

22. If A and B are square matrices of order 3 such that $|A| = -1$ and $|B| = 3$, then $|3AB|$ is equal to

(a) -9 (b) -27

(c) -81 (d) 81

23. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is

(a) idempotent (b) unit

(c) unitary (d) nilpotent

24. The inverse of the matrix $\begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$ is **[CE GATE 2010]**

(a) $\frac{1}{12}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (b) $\frac{1}{12}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

(c) $\frac{1}{14}\begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ (d) $\frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

25. For a matrix $[M] = \begin{bmatrix} 3/5 & 4/5 \\ x & 3/5 \end{bmatrix}$, the transpose of the matrix is equal to the inverse of the matrix $[M]' = [M]^{-1}$. The value of x is **[ME GATE 2009]**

(a) $-\frac{4}{5}$ (b) $-\frac{3}{5}$ (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

26. A is $m \times n$ full rank matrix with $m > n$ and I is an identity matrix. Let matrix $A' = (A')^{-1} \cdot A'$. Then which one of the following statements is false? **[EE GATE 2008]**

(a) $AA'A = A$ (b) $(AA')^2$

(c) $A'A = I$ (d) $AA'A = A$

27. Let A, B, C and D be $n \times n$ matrices, each with non-zero determinant. If $ABCD = I$, then B^{-1} is **[CS GATE 2004]**

(a) $D^{-1}C^{-1}A^{-1}$

(b) CDA

(c) ADC

(d) Does not necessarily exist

28. Let A be the matrix of order $m \times n$, then the determinant of A exists iff

(a) $m > n$ (b) $m < n$

(c) $m \neq n$ (d) $m = n$

29. If matrices A and B commute, then

(a) $(AB)^n = A^n B^n$ (b) $(AB)^n = AB$

(c) $(AB)^n = B^n$ (d) None of these

30. If I is an identity matrix, then

(a) $I^n = I$ (b) $I^n = 0$

(c) $I^n = 1/I$ (d) None of these

31. If A and B are two matrices of same order, then the following operation does not hold

(a) $A + B = B + A$ (b) $AB = BA$

(c) $A - B = -B + A$ (d) $(A + B)I = A + B$

32. If A is a square matrix, then A^{-1} exists iff

(a) $|A| = 0$ (b) $|A| \neq 0$

(c) $|A| > 0$ (d) $|A| < 0$

33. The matrix $A = [a_{ij}]$ is Hermitian iff

(a) $a_{ij} = -\bar{a}_{ji}$ for all i, j (b) $a_{ij} = \bar{a}_{ji}$ for all i, j

(c) $a_{ij} = a_{ji}$ for all i, j (d) None of these

34. The diagonal elements of Hermitian matrix are

(a) complex numbers (b) real numbers

(c) natural numbers (d) None of these

35. The diagonal elements of Skew-Hermitian matrix are

(a) pure real numbers or zero

(b) pure imaginary or zero

(c) complex numbers

(d) None of the above

36. The matrix $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ is a

(a) Hermitian matrix

(b) skew-Hermitian matrix

(c) symmetric matrix

(d) skew-symmetric matrix

37. The matrix $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is a

(a) Hermitian matrix (b) skew-Hermitian

(c) skew-symmetric (d) symmetric

Answers with Solutions

1. (b) **2.** (d) **3.** (c) **4.** (c) **5.** (c) **6.** (b)

7. (a) $\because\ A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$$

Hence, A is idempotent matrix.

8. (b) **9.** (c) **10.** (a) **11.** (a) **12.** (a) **13.** (a)
14. (c) **15.** (d) **16.** (c)

17. (c) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Rightarrow A' = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

$$\Rightarrow AA' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$$

So, $(AA')^{-1} = \dfrac{\text{adj}(A)}{|A|} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

18. (d) $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow A^2 = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 2a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3a \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = A^3 \cdot A = \begin{bmatrix} 1 & 3a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4a \\ 0 & 1 \end{bmatrix}$$

19. (c)

20. (a) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{(3-0)}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

So, $(A^{-1})^2 = \dfrac{1}{9}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}^2$

$$= \frac{1}{9}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$(A^{-1})^2 = \frac{1}{9}\begin{bmatrix} 1 & -8 \\ 0 & 9 \end{bmatrix}$$

$$\Rightarrow (A^{-1})^3 = (A^{-1})^2 \cdot A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & -8 \\ 0 & 9 \end{bmatrix} \times \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow (A^{-1})^3 = \frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$$

21. (c) **22.** (c) **23.** (b)

24. (d) $A = \begin{bmatrix} 3+2i & i \\ -i & 3-2i \end{bmatrix}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So, $A^{-1} = \dfrac{1}{(3+2i)(3-2i)+i^2}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3-2i & -i \\ i & 3+2i \end{bmatrix}$$

25. (a) $\because\ M' = M^{-1}$

or $\begin{bmatrix} 3/5 & x \\ 4/5 & 3/5 \end{bmatrix} = \begin{bmatrix} 3/5 & -4/5 \\ -x & +3/5 \end{bmatrix}$

$$\Rightarrow x = -\frac{4}{5}$$

26. (b) **27.** (b)

28. (d) The determinant exists only for square matrix
$\therefore\ m = n$

29. (a) By induction,

$$(AB)^2 = AB \cdot AB = AA\ BB = A^2B^2$$
$$(AB)^3 = (AB)^2 AB = A^2B^2AB$$
$$= A^2AB^2B = A^3B^3 : (AB)^{n-1} = A^{n-1}B^{n-1}$$
$$(AB)^n = (AB)^{n-1}AB = A^{n-1}B^{n-1}AB$$
$$= A^{n-1}AB^{n-1}B = A^nB^n$$

30. (a) I in an identity matrix $\Rightarrow I^2 = I$

$$I^3 = I^2 \cdot I = I \cdot I = I : I^{n-1} = I$$
$$I^n = I^{n-1} \cdot I = I \cdot I = I$$

31. (b) Matrix multiplication is not commutative operation

$Q(X) = X^T AX$ is positive definite

$\Leftrightarrow \quad Q(X) \geq 0, X \neq 0$

$\Leftrightarrow$ All eigen values of A, $\lambda > 0$

32. (b) A^{-1} exist $\Leftrightarrow$ A is non-singular $\Leftrightarrow$ $|A| \neq 0$.

33. (b) $A = [a_{ij}]$ is Hermitian $\Rightarrow A^{-T} = A$, *i.e.*, $a_{ij} = \bar{a}_{ji}$

34. (b) A is Hermitian $\Rightarrow a_{ij} = \bar{a}_{ji}$,

$\Rightarrow a_{ii} = \bar{a}_{ii} \quad \Rightarrow \quad a_{ij}$ are real numbers.

35. (b) A is skew-Hermitian

$\Rightarrow \quad \bar{a}_{kj} = -a_{jk}$

$\Rightarrow \quad \bar{a}_{jj} = -a_{ij} \quad \Rightarrow \quad a_{ij}$

are pure imaginary or zero, *i.e.*, if $a_{ij} = k_i$

$\Rightarrow \quad \bar{a}_{ij} = -k_i, a_{ij} = 0 \quad \Rightarrow \quad \bar{a}_{ij} = 0$

36. (a) $A = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ and $\bar{A} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

and $\quad A^\theta = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} = A$

$\because \quad A^\theta = A$

So, A is Hermitian.

37. (b) $A = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$\Rightarrow \quad \bar{A} = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$

and $\quad A^\theta = (\bar{A})' = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = -A$

So, A is skew-Hermitian.

Rank of a Matrix

Definition

A positive number r is said to be the rank of given matrix 'A' of order $m \times n$, if it satisfies the following properties :

- **(i)** There is atleast one square submatrix of order r whose determinant is not equal to zero.
- **(ii)** The determinant of order higher than r, *i.e.*, $(r+1)$ should be zero.

The rank of matrix is denoted by $\rho(A)$ or $r(A)$.

The most useful applicatioin of finding the rank of a matrix is the computation of the number of solutions of a system of linear equations.

- From the definition of the rank of a matrix A of order $m \times n$. We conclude that
  - (i) If the matrix A does not possess any minor of order $(r+1)$, then $\rho(A) \leq r$.
  - (ii) If atleast one minor of order r of the matrix is not equal to zero, then $\rho(A) \geq r$.
- If every $(r+1)$th order minor of the matrix is zero, then any higher order minor will also be zero.
- The rank of a null matrix is defined as zero *i.e.*, $\rho(O) = 0$.
- If I_n is a unit matrix of order n, then its rank is n *i.e.*, $\rho(I_n) = n$.
- If A is $n \times n$ non-singular matrix, then $\rho(A) = n$.

Properties of Rank

(a) Rank of a matrix is same as the number of linearly independent row vectors in the matrix as well as the number of linearly independent column vectors in the matrix.

(b) The rank of a matrix A does not change by pre-multiplication or post-multiplication with any non-singular matrix.

(c) If A and B are matrices of same order, then

$$\rho(A+B) \leq \rho(A) + \rho(B)$$

(d) If A and B are matrices of same order, then

$$\rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

where, A and B are conformable for multiplication.

Thus, $\quad \rho(AB) \leq \rho(A)$

and $\quad \rho(AB) \leq \rho(B)$

(e) If A' is a transpose of matrix A, then

$$\rho(A') = \rho(A) \quad \text{and} \quad \rho(AA') = \rho(A)$$

(f) If A^θ is the conjugate transpose of A, then

$$\rho(A^\theta) = \rho(A) \quad \text{and} \quad \rho(AA^\theta) = \rho(A)$$

(g) If A is a matrix of order $m \times n$, then

$$\rho(A) \leq \min\{m, n\}$$

Thus, $\quad \rho(A) \leq m \quad$ and $\quad \rho(A) \leq n$

(h) The rank of a skew-symmetric matrix cannot be one.

(i) If A and B are two n-rowed square matrices, then

$$\rho(AB) \geq \rho(A) + \rho(B) - n$$

Elementary Operations

An elementary operation is said to be row (or column) operation if it is applied to rows (or columns). There are three types of elementary operations given below :

(i) The interchange of any two rows (columns) *i.e.,* the interchanging of *i*th and *j*th rows (or columns) of the matrix. This operation is denoted by $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$) or by R_{ij} (or C_{ij}).

(ii) The multiplication of any row (column) by a non-zero number *i.e.,* if *i*th row (or column) of the matrix is multiplied by scalar k ($k \neq 0$), then this row (or column) operation is denoted by $R_i \to kR_j$ (or $C_i \to kC_j$).

(iii) The addition of constant multiple of the elements of any row (column) to the corresponding elements of any other row (column) *i.e.,* if k times of the elements of the *i*th row (or column) are added to the corresponding elements of *j*th row (or column), then this row (or column) operation is denoted by $R_j \to R_j + kR_i$ (or $C_j \to C_j + kC_i$).

Elementary Matrices

An elementary matrix is that, which is obtained from a unit matrix by subjecting it to any of the elementary operations.

There are three types of the elementary matrices :

(i) The elementary matrices which are obtained by applying elementary operations $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$ on unit matrix are same and denoted by E_{ij}.

Example 1. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $E_{12} - E_{21}$ is equal to

(a) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$ (d) None of these

Sol. (b) We have,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

and $$E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\therefore \quad E_{12} - E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(ii) The elementary matrices which are obtained by applying elementary operations $R_i \to kR_i$ or $C_i \to kC_i$ on the unit matrix are same and denoted by $E_i(k)$.

(iii) The elementary matrices which are obtained by applying elementary operations $R_j \to R_j + kR_i$ and $C_i \to C_i + kC_j$ on unit matrix are same and denoted by $E_{ji}(k)$.

Example 2. If $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then $2E_{12}(2) + 3E_{21}$ is equal to

(a) $\begin{bmatrix} 2 & 7 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 7 & 2 \\ 2 & 3 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 7 \\ 3 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 4 \\ 3 & 1 \end{bmatrix}$

Sol. (c) We have,

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad E_{12}(2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad R_1 \to R_1 + 2R_2$$

$$E_{21} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad R_1 \longleftrightarrow R_2$$

$$\therefore \quad 2E_{12}(2) + 3E_{21} = 2\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + 3\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 3 & 2 \end{bmatrix}$$

Properties of Elementary Operations (Matrices)

(a) The matrix obtained by applying a row operation on a given matrix be same as the matrix obtained by pre-multiplication of the given matrix by the corresponding elementary matrix.

(b) The matrix obtained by applying a column operation on a given matrix be same as the matrix obtained by post-multiplication of the given matrix by the corresponding elementary matrix.

e.g., Let $A = \begin{bmatrix} 3 & 4 \\ 8 & 9 \end{bmatrix}$, then by operating R_{12}, we get

$$A \sim \begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix} \quad \ldots\text{(i)}$$

Now, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then by operating R_{12}, we get

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \quad E_{12} \cdot A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 8 & 9 \end{bmatrix} = \begin{bmatrix} 8 & 9 \\ 3 & 4 \end{bmatrix}$$

which is same as (i).

(c) The effect of row operation on the product of two matrices is equivalent to the effect of the same row operation applied only to the pre-factor.

(d) The effect of a column operation on the product of two matrices is equivalent to the effect of the same column operation applied only to the post-factor.

(e) Elementary operations do not change the rank of a matrix.

Equivalent Matrix

Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary operations. The symbol ~ is used for equivalence.

Two equivalent matrices have the same order and the same rank.

Echelon Form or Triangular Form

A matrix is said to be in Echelon form if

(i) All the non-zero rows, if any precede the zero rows.

(ii) The number of zeros preceeding the Ist non-zero element in a row is less than that the number of such zeros in the next row.

(iii) The first non-zero element in every row is unity, *i.e.,* the elements of main diagonal must be unit if possible.

(iv) The number of non-zero rows is the rank of given matrix.

Example 3. Given matrix $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

The rank of the matrix is **[CE GATE 2003]**

(a) 4 (b) 3

(c) 2 (d) 1

Sol. (c) ∵ We have,

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Performing $R_1 \to R_1 \div 4$

$$A = \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$$

Performing $R_2 \to R_2 - 6R_1, R_3 \to R_3 - 2R_1$

$$A = \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 5/2 & 5/2 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix}$$

Now, performing $R_2 \to R_2 \times \left(\frac{2}{5}\right)$

$$A = \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1/2 & -1/2 \end{bmatrix}$$

Performing $R_3 \to R_3 + \frac{1}{2} R_2$

$$A = \begin{bmatrix} 1 & 1/2 & 1/4 & 3/4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence, number of non-zero rows = 2

So, rank of given matrix = 2

Normal Form or Canonical Form

On performing elementary row and column transformation, any non-zero matrix of $m \times n$ order can be reduced to one of the following forms called the normal form.

(i) I_r **(ii)** $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$

(iii) $[I_r, 0]$ **(iv)** $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The number r so obtained from above is called the rank of given matrix.

To reduce the matrix A to the normal form, apply row and column operations on A. So as to make the diagonal elements of A equal to 1 as for as possible after which the diagonal elements, if exist, are all zero and non-diagonal elements are made zeros.

Example 4. Reduce the following matrix to the normal form

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

and find the rank of A.

(a) 1 (b) 2

(c) 3 (d) 0

Sol. (c) Given,

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

Applying row operations $R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$ on A, we get

$$A \sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

Applying the column operations $C_2 \to C_2 - 2C_1$, $C_3 \to C_3 + C_1, C_4 \to C_4 - 2C_1, C_5 \to C_5 - C_1$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix}$$

Applying $C_2 \longleftrightarrow C_5$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 3 & -6 & 0 \\ 0 & 2 & 5 & -12 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 - 2R_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -6 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Applying $C_3 \to C_3 - 3C_2, C_4 \to C_4 + 6C_2$, we get

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{bmatrix}$$

Applying $R_3 \to (-1)\, R_3$, we get

$$A \sim \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$= [I_3 \quad O]$$

which is normal form of A, where O is the null matrix of order 3×2. Hence,

$$\text{Rank } A = 3$$

Example 5. $X = [X_1\ X_2 \ldots X_n]'$ is an n-tuple non-zero vector. The $n \times n$ matrix $V = XX'$ **[EE GATE 2007]**

(a) has rank zero (b) has rank 1

(c) is orthogonal (d) has rank n

Sol. (b) $\because\ X = [X_1\ X_2 \ldots X_n]'$

$$V = XX' = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} [X_1\ X_2 \ldots X_n]$$

$$= \begin{bmatrix} X_1^2 & X_1X_2 & X_1X_3 & \ldots & X_1X_n \\ X_2X_1 & X_2^2 & X_2X_3 & \ldots & X_2X_n \\ X_3X_1 & X_3X_2 & X_3^2 & \ldots & X_3X_n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ X_nX_1 & X_nX_2 & X_nX_3 & \ldots & X_n^2 \end{bmatrix}$$

Now, if we perform following elementary operations on above $(\because X_1, X_2, \ldots, X_n \neq 0)$

$$R_2 \to \frac{R_2}{x_2} - \frac{R_1}{x_1}$$

$$R_3 \to \frac{R_3}{x_3} - \frac{R_1}{x_1}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$R_n \to \frac{R_n}{x_n} - \frac{R_1}{x_1}$$

We get
$$\begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 & \ldots & x_1x_n \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

$\because \quad x_1^2 \neq 0$

Hence, rank of matrix = 1

Intro Exercise 2

1. The rank of the marix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is **[EC GATE 2006]**

(a) zero (b) 1 (c) 2 (d) 3

2. The rank of the matrix $\begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) zero

3. The rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) None of these

4. The rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is, when the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear

(a) greater than 3 (b) less than 3
(c) greater than 4 (d) None of the above

5. The rank of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is

(a) 1 (b) 2
(c) 3 (d) zero

6. The rank of the diagonal matrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ is}$$

(a) 1 (b) 2 (c) 4 (d) 3

7. The rank of the matrix $\begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \\ 3 & -6 & 9 \end{bmatrix}$ is

(a) 3 (b) 2 (c) 0 (d) 1

8. If A is a non-zero column vector $(n \times 1)$, then the rank of matrix AA' is

(a) zero (b) 1 (c) $n-1$ (d) n

9. The rank of matrix $\begin{bmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ -1 & 0 & \mu \end{bmatrix}$ is 2, then μ is equal to

(a) any row number (b) 3
(c) 1 (d) 2

10. If P and Q are non-singular matrices, then for matrix M, which of the following is correct?

(a) Rank $(PMQ) >$ Rank M
(b) Rank $(PMQ) =$ Rank M
(c) Rank $(PMQ) <$ Rank M
(d) Rank $(PMQ) =$ Rank M + Rank (PQ)

11. Rank of singular matrix of order 4 can be at most

(a) 1 (b) 2
(c) 3 (d) 4

12. If the rank of an $n \times n$ matrix A is $(n-1)$, then the system of equations $Ax = b$ has

(a) $(n-1)$ parameter family of solutions
(b) one parameter family of solutions
(c) no solution
(d) a unique solution

13. The rank of matrix $\begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$ is 3, then the value of μ is

(a) zero (b) 1
(c) 4 (d) –1

14. The rank of the matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is

(a) 4 (b) 3
(c) 1 (d) zero

15. Let M be a $m \times n \, (m < n)$ matrix with rank m, then

(a) for every b in R^m, $Mx = b$ has unique solution
(b) for every b in R^m, $Mx = b$ has a solution but it is not unique
(c) there exists $b \in R^m$ for which $Mx = b$ has no solution
(d) None of the above

16. Let $M = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, then the rank of M is equal to

(a) 3 (b) 4 (c) 2 (d) 1

17. Let A be a matrix of order $m \times n$ and R is non-singular matrix of order n, then

(a) $\text{rank}(RA) \neq \text{rank}(A)$ (b) $\text{rank}(RA) \geq \text{rank}(A)$

(c) $\text{rank}(RA) \leq \text{rank}(A)$ (d) $\text{rank}(RA) = \text{rank}(A)$

18. Given $A\bar{x} = \bar{b}$, then the solution of this system exists, if

(a) $\text{rank}(A) \neq \text{rank}[A, b]$

(b) $\text{rank}(A) = \text{rank}(b)$

(c) $\text{rank}(A) = \text{rank}[A; b]$

(d) None of these

19. For given $A\bar{x} = \bar{b}$, where order of A is n we have unique solution, if

(a) $\text{rank}(A) \neq \text{rank}[A; b] = n$

(b) $\text{rank}(A) = \text{rank}[A; b] \neq n$

(c) $\text{rank}(A) = \text{rank}[A; b] = n$

(d) None of the above

20. If A is a $(n \times 1)$ non-zero matrix and B is $(1 \times n)$ non-zero matrix, then

(a) $\text{rank}(AB) = 1$ (b) $\text{rank}(AB) = n$

(c) $\text{rank}(AB) = 0$ (d) None of these

21. The rank of the matrix $A = \begin{bmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) 4

22. The rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{bmatrix}$ is

(a) 1 (b) 2 (c) 3 (d) 4

23. The rank of matrix, whose every element is unity, is

(a) greater than one (b) equal to one

(c) zero (d) None of these

24. Let A be a square matrix of order n, then nullity of A is

(a) $n - \text{rank}\, A$ (b) $\text{rank}\, A - n$

(c) $n + \text{rank}\, A$ (d) None of these

25. If I is an unit matrix of order n, then

(a) $\text{rank}(I) = n$ (b) $\text{rank}(I) \geq n$

(c) $\text{rank}(I) \geq n$ (d) None of these

Answers with Solutions

1. (c) $\because |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$

$= 1(-1-0) - 1(1-0) + 1(1+1)$

$= -1 - 1 + 2 = 0$

$\therefore \quad \rho(A) < 3$

Now, since $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$

$\therefore \quad \rho(A) = 2$

2. (c) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 1 & 5 & 4 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 3 & 2 \\ 0 & 4 & 1 \\ 0 & -2 & -2 \end{bmatrix} \quad \begin{pmatrix} R_2 \to 2R_1 - R_2 \\ R_3 \to R_1 - R_3 \end{pmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 2 & 2 \end{bmatrix} \quad \begin{pmatrix} C_2 \to 3C_1 - C_2 \\ C_3 \to 2C_1 - C_3 \end{pmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 2 & 2 \end{bmatrix} \quad \left(R_2 \to \frac{R_2}{-4}\right)$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 3/2 \end{bmatrix} \quad (R_3 \to 2R_2 - R_3)$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3/2 \end{bmatrix} \quad \left(C_3 \to \frac{C_2}{4} - C_3\right)$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \left(C_3 \to \frac{C_3}{-3/2}\right)$

which is in normal form.

Hence, rank $(A) = 3$

3. (a) Given, $A = \begin{bmatrix} 1 & 1 & 1 \\ a & a & a \\ a^3 & a^3 & a^3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} R_2 \to aR_1 - R_2 \\ R_3 \to a^3R_1 - R_3 \end{pmatrix}$

which is in echelon form.

Hence, $\rho(A) = 1$

4. (b) Given, $A = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$

$\therefore \quad |A| = x_1(y_2 - y_3) - y_1(x_2 - x_3) + 1(x_2y_3 - y_2x_3)$
$= 0$

If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

$\therefore \quad \rho(A) < 3$

5. (c) $\because$ The matrix has all diagonal elements non-zero, so rank = number of rows = 3.

6. (d) $\because$ The diagonal matrix has three non-zero rows. Hence, rank is 3.

7. (a) Given, $A = \begin{bmatrix} 2 & -4 & 6 \\ -1 & 2 & -3 \\ 3 & -6 & 9 \end{bmatrix}$

$$\sim \begin{bmatrix} 2 & -4 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{pmatrix} R_2 \to \frac{R_1}{2} + R_2 \\ R_3 \to \frac{3}{2}R_1 - R_3 \end{pmatrix}$$

$\Rightarrow \quad \rho(A) = 3$

8. (a)

9. (c) The rank to be 1, any one row must be zero. It is possible only when

$\mu - 1 = 0 \quad \Rightarrow \quad \mu = 1$

10. (b) **11.** (c) **12.** (b) **13.** (a) **14.** (a)

15. (b) **16.** (c) **17.** (d) **18.** (c) **19.** (c) **20.** (a)

21. (b) $\because \quad |A| = 0$

and $\begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix} \neq 0$

So, rank of $A = 2$

22. (c) $\because \quad |a_{11}| = 1 \neq 0$

$\Rightarrow \quad \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3 \neq 0$

$\Rightarrow \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 1 & 2 \end{vmatrix} \neq 0$

So, rank of $A = 3$

23. (b) $\because \quad |A| = 1 \neq 0$

and $111 = 1 \neq 0$

$\Rightarrow \quad \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$

and $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$

So, rank of $A = 1$

24. (a) $\because$ Nullity = Order of matrix − Rank of matrix
$= n - \text{rank of } A$

25. (a)

System of Linear Equations

Let us consider the following system of m linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m$$

This system of equations can be written as

$$AX = B$$

where, $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = [x_1, x_2, \ldots, x_n]^T$$

and $$B = [b_1, b_2, \ldots, b_m]^T$$

The matrix A is known as coefficient matrix of the system of equations.

Solution of System

A set of values of $x_1, x_2, \ldots, x_n$, which simultaneously satisfy all the equations, is known as a solution of system of equations.

Consistent and Inconsistent System

If the system of equations has one or more solutions. Then it is said to be consistent systems otherwise it is said to be inconsistent system of equations.

Homogeneous and Non-homogeneous System

A system of linear equations $AX = B$ is called homogeneous system if $B = 0$ and non-homogeneous system of equations if $B \neq 0$.

Non-homogeneous System of Linear Equations

There are various methods for solving a system of non-homogeneous linear equations *viz*

(i) Cramer's rule

(ii) Matrix method

(iii) Rank method

We have already discussed the Cramer's rule in previous classes.

Matrix Method

This method is applicable when the number of equations is same as the number of unknowns.

Suppose the system of n linear equations in n unknowns is given by

$$AX = B$$

Then

***Case* I.** When $|A| \neq 0$ *i.e.,* coefficient matrix A is non-singular. Then, A^{-1} exists.

$$\therefore \quad AX = B$$

$$\Rightarrow \quad A^{-1}(AX) = A^{-1}B$$

$$\Rightarrow \quad X = A^{-1}B$$

which is the solution of the given system of equations. Now, let X_1 and X_2 be two solutions of the given system of equations. Therefore,

$$AX_1 = B \quad \text{and} \quad AX_2 = B$$

$$\Rightarrow \quad AX_1 = AX_2$$

$$\Rightarrow \quad A^{-1}(AX_1) = A^{-1}(AX_2) \qquad (\because A^{-1} \text{ exists})$$

$$\Rightarrow \quad (A^{-1}A)X_1 = (A^{-1}A)X_2$$

$$\Rightarrow \quad IX_1 = IX_2$$

$$\Rightarrow \quad X_1 = X_2$$

Thus, in this case the system $AX = B$ is consistent and has unique solution, given by

$$X = A^{-1}B$$

***Case* II.** When $|A| = 0$ and $(\text{adj}A)B = 0$. Then, the system is consistent and has infinitely many solutions.

***Case* III.** When $|A| = 0$ and $(\text{adj}A)B \neq 0$. Then, the system is inconsistent.

Rank Method

This is general method for solving a system of m linear equations in n unknowns given by

$$AX = B$$

where,
$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} \end{bmatrix}_{m \times n}$$

$$X = [x_1, x_2, \ldots, x_n]^T$$

$$B = [b_1, b_2, \ldots, b_m]^T$$

The matrix

$$\overline{A} = [A : B] = \begin{bmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & \vdots & b_1 \\ a_{21} & a_{22} & \cdots\cdots & a_{2n} & \vdots & b_2 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \vdots & \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \vdots & \cdots\cdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & \vdots & b_m \end{bmatrix}$$

is known as augmented matrix of given system of equations.

This system of linear equations is consistent if the rank of augmented matrix $[A : B]$ is same as to the rank of coefficient matrix A.

***Case* I.** When $m > n$, then

(a) If $\rho(A) = \rho[A : B] = n$, then the system has unique solution.

(b) If $\rho(A) = \rho[A : B] = r < n$, then the system has infinite number of solutions. In this case $(n-1)$ variable can be assigned arbitrary values.

(c) If $\rho(A) \neq \rho[A : B]$, then the system of linear equations is inconsistent.

***Case* II.** When $m < n$ and $\rho(A) = \rho[A : B] = r \leq m < n$, then the system has infinite number of solutions.

Example 1. Solution for the system defined by the set of non-homogeneous lines as equations

$$x + y + z = 6$$
$$x + 2y + 3z = 14$$
$$x + 4y + 7z = 30 \text{ is}$$

(a) $x = k - 2,\ y = 8 - 2k,\ z = k$

(b) $x = 2,\ y = -1,\ z = 3$

(c) $x = 0,\ y = k,\ z = 4$

(d) Not exist

Sol. (a) Putting the given system in the form

$$AX = B$$

where,
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$B = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

So, augmented matrix $\tilde{A} = [A : B]$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, rank of $[A : B] = 2$

Rank of $[A] = 2 < 3$ $(n = 3)$

i.e., $n - r = 3 - 2 = 1$

So, the system is consistent and has infinite solution.

So from $[A]X = [B]$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $x + y + z = 6$ ...(i)

$y + 2z = 8$ ...(ii)

Let $z = k$

From Eq. (ii),

$\Rightarrow$ $y = 8 - 2k$

Putting y and z in Eq. (i),

We get $x = k - 2$

Hence, $x = k - 2$, $y = 8 - 2k$, $z = k$.

Homogeneous System of Equations

There are two methods given below :

(i) Matrix method

(ii) Rank method

(i) Matrix Method

Suppose the system of n linear equations in n unknowns is given by

$$AX = O$$

where,
$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} \end{bmatrix}$$

$$X = [x_1, x_2, \ldots, x_n]^T$$

$$O = [0, 0, 0, \ldots, 0]^T$$

***Case* I.** When $|A| \neq 0$ *i.e.,* A is non-singular, then A^{-1} exists.

$\therefore$ $AX = O$

$\Rightarrow$ $A^{-1}(AX) = A^{-1}O$

$\Rightarrow$ $(A^{-1}A)X = O$

$\Rightarrow$ $X = O$

Thus, in this case the system is consistent and has unique solution given by $X = 0$ *i.e.,* $x_1 = x_2 = x_3 = \ldots = x_n = 0$. This solution is known as trivial solution.

***Case* II.** When $|A| = 0$. Then, the system has infinitely many non-trivial solutions and these solutions are obtained as follows :

If $\rho(A) = r$, then putting $(n - r)$ variable equal to arbitrary constant k (any real number) and values of remaining variables are obtained to get infinitely many solutions of the system.

(ii) Rank Method

In this case the rank of augmented matrix is always equal to the rank of coefficient matrix. So, a homogeneous system of equations is always consistent.

Case I. If $\rho(A) = n =$ number of variables, then the system $AX = O$ has unique solution $X = O$ *i.e.,* $x_1 = x_2 = x_3 = \ldots = x_n = 0$ which is trivial solution.

Case II. If $\rho(A) = r < n$ (= number of variables), then the system of equations has an infinite number of solutions.

Example 2. Solution for the system defined by the set of equations

$$x + 2y + 3z = 0$$
$$3x + 4y + 4z = 0$$
$$7x + 10y + 12z = 0 \text{ is}$$

(a) $x = y = z = 0$ (b) $x = y = 0, z = 1$
(c) $x = 1, y = z = 0$ (d) $x = 1, y = -2, z = 1$

Sol. (a) $\because$ Putting equations in the matrix form

$$AX = B$$

We get, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Now, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

Operating $R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 7R_1$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & -4 & -9 \end{bmatrix}$$

Operating $R_3 \to R_3 - 2R_2$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & -1 \end{bmatrix}$$

Operating $R_2 \to R_2 \div (-2)$

$R_3 \to R_3 \div (-1)$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Number of non-zero rows = 3

So, rank = 3 = number of unknowns

Hence, the given system of equations has trivial solution

$$x = y = z = 0$$

Example 3. The following system of homogeneous equations

$$2x + y + 2z = 0$$
$$x + y + 3z = 0$$
$$4x + 3y + bz = 0$$

has non-trivial solution then value of b is

(a) $b = 1$ (b) $b = 2$
(c) $b = 4$ (d) $b = 8$

Sol. (d) ∵ The system of homogeneous equations has non-trivial solution.

Hence, $|A| = 0$

$$\Rightarrow \begin{vmatrix} 2 & 1 & 2 \\ 1 & 1 & 3 \\ 4 & 3 & b \end{vmatrix} = 0$$

$$\Rightarrow 2\begin{vmatrix} 1 & 3 \\ 3 & b \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 4 & b \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} = 0$$

$$2(b-9) - 1(b-12) + 2(3-4) = 0$$

$$\Rightarrow b - 8 = 0$$

$$\Rightarrow b = 8$$

Example 4. Consider a non-homogeneous system of linear equations representing mathematically an over determined system. Such a system will be **[CE GATE 2005]**

(a) consistent having a unique solution
(b) consistent having many solutions
(c) inconsistent having a unique solution
(d) inconsistent having no solution

Sol. (a), (b) and (d) all are possible.

In an over determined system having more equations than variables all three possibilities can exist :

(a) Consistent and unique if $r = n$
(b) Consistent and infinite if Rank of $\tilde{A}$ = Rank of $A \neq$ number of unknowns
(c) inconsistent and no solution if Rank of $[A : B] \neq$ Rank of $[A]$.

Example 5. For what values of α and β, the following simultaneous equations have an infinite number of solution? **[CE GATE 2007]**

$$x + y + z = 5$$
$$x + 3y + 3z = 9$$
$$x + 2y + \alpha z = \beta$$

(a) 2, 7 (b) 3, 8
(c) 8, 3 (d) 7, 2

Sol. (a) Putting the system of simultaneous equations in the form

$$AX = B$$

We get, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 2 & \alpha \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 5 \\ 9 \\ \beta \end{bmatrix}$

So, the augmented matrix

$$\tilde{A} = [A \vdots B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 5 \\ 1 & 3 & 3 & \vdots & 9 \\ 1 & 2 & \alpha & \vdots & \beta \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & 2 & 2 & \vdots & 4 \\ 0 & 1 & \alpha - 1 & \vdots & \beta - 5 \end{bmatrix}$$

Operating $R_2 \to R_2 \div 2$

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & 1 & 1 & \vdots & 2 \\ 0 & 1 & \alpha - 1 & \vdots & \beta - 5 \end{bmatrix}$$

Operating $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & \vdots & 5 \\ 0 & 1 & 1 & \vdots & 2 \\ 0 & 0 & \alpha - 2 & \vdots & \beta - 7 \end{bmatrix}$$

Now for infinite solution the last row must be zero.

So, $\alpha - 2 = 0 \Rightarrow \alpha = 2$

and $\beta - 7 = 0 \Rightarrow \beta = 7$

Example 6. The following simultaneous equations

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + kz = 6$$

will not have a unique solution for R is equal to **[CE GATE 2008]**

(a) 0 (b) 5
(c) 6 (d) 7

Sol. (d) Putting the given system in $AX = B$ form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & k \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Now, augmented matrix

$$\tilde{A} = [A \vdots B] = \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 1 & 2 & 3 & \vdots & 4 \\ 1 & 4 & k & \vdots & 5 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 3 & k-1 & \vdots & 3 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$= \begin{bmatrix} 1 & 1 & 1 & \vdots & 3 \\ 0 & 1 & 2 & \vdots & 1 \\ 0 & 0 & k-7 & \vdots & 0 \end{bmatrix}$$

Now, if $k - 7 = 0, k = 7$

i.e., rank of $[A]$ = rank of $[A \vdots B] = 2 \neq 3$ (Number of unknowns)

So, the system of simultaneous equations has infinite solution.

Example 7. For what value of 'a' if any equation of the following system of equations in x, y, z has a solution? **[ME GATE 2008]**

$$2x + 3y = 4$$
$$x + y + z = 4$$
$$x + 2y - z = a$$

(a) Any real number (b) 0
(c) 1 (d) There is no such value

Sol. (b) Putting the system in $AX = B$ form

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 4 \\ 4 \\ a \end{bmatrix}$$

So, augmented matrix

$$\tilde{A} = [A : B] = \begin{bmatrix} 2 & 3 & 0 & \vdots & 4 \\ 1 & 1 & 1 & \vdots & 4 \\ 1 & 2 & -1 & \vdots & a \end{bmatrix}$$

Operating $R_1 \to R_1 \div 2$

$$= \begin{bmatrix} 1 & 3/2 & 0 & \vdots & 2 \\ 1 & 1 & 1 & \vdots & 4 \\ 1 & 2 & -1 & \vdots & a \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 3/2 & 0 & \vdots & 2 \\ 0 & -1/2 & 1 & \vdots & 2 \\ 0 & 1/2 & -1 & \vdots & a-2 \end{bmatrix}$$

Operating $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 3/2 & 0 & \vdots & 2 \\ 0 & -1/2 & 1 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & a \end{bmatrix}$$

Now, the following cases arise :

(i) If $a = 0$, then Rank $[A] = 2$,
Rank $[A : B] = 2$
Then the system will be consistent and will have infinite solution.

(ii) If $a \neq 0$, then Rank $[A] = 2$
Rank $[A \vdots B] = 3$
$\because$ Rank $[A] \neq$ Rank $[A \vdots B]$
So, the system of linear equations will have no solution.

Example 8. In the matrix equation $PX = q$, which of the following is a necessary condition for the existence of atleast one solution for the unknown vector? **[EE GATE 2005]**

(a) Augmented matrix $[pq]$ must have the same rank as matrix p
(b) Vector q must have only non-zero elements
(c) Matrix p must be singular
(d) Matrix p must be square

Sol. (a) $\because$ Rank $[pq]$ = Rank $[p]$ is necessary for existence of atleast one solution.

Example 9. Consider the following system of linear equations :

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and the third columns of the coefficient matrix are linearly dependent. For how many values of α does the system of equations have infinitely many solutions? **[CS GATE 2003]**

(a) 0 (b) 1
(c) 2 (d) Infinitely many

Sol. (b)

We have, $A = \begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$

So, augmented matrix

$$[A \vdots B] = \begin{bmatrix} 2 & 1 & -4 & \vdots & \alpha \\ 4 & 3 & -12 & \vdots & 5 \\ 1 & 2 & -8 & \vdots & 7 \end{bmatrix}$$

Operating $R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - \frac{1}{2} R_1$

$$= \begin{bmatrix} 2 & 1 & -4 & \vdots & \alpha \\ 0 & 1 & -4 & \vdots & 5-2\alpha \\ 0 & 3/2 & -6 & \vdots & 7-\alpha/2 \end{bmatrix}$$

Operating $R_3 \to R_3 - \frac{3}{2} R_2$

$$= \begin{bmatrix} 2 & 1 & -4 & \vdots & \alpha \\ 0 & 1 & -4 & \vdots & 5-2\alpha \\ 0 & 0 & 0 & \vdots & \frac{5\alpha - 1}{2} \end{bmatrix}$$

For infinite many solutions there is a necessary condition that atleast one row must be zero.

So, for this $\frac{5\alpha - 1}{2} = 0$

$\Rightarrow$ $\alpha = \frac{1}{5}$

So, α contains only 1 value.

Example 10. The following system of equations

$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 2$$
$$x_1 + 4x_2 + ax_3 = 4$$

has a unique solution. The only possible value of a is/are **[CS GATE 2008]**

(a) 0
(b) either 0 or 1
(c) one of 0, 1 and – 1
(d) any real number other than 5

Sol. (d) Putting the given linear equations in $AX = B$ form

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 3 \\ 1 & 4 & a \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

Now, augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 1 & 2 & 3 & \vdots & 2 \\ 1 & 4 & a & \vdots & 4 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$= \begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 3 & a-2 & \vdots & 3 \end{bmatrix}$$

Operating $R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & 1 & 2 & \vdots & 1 \\ 0 & 1 & 1 & \vdots & 1 \\ 0 & 0 & a-5 & \vdots & 0 \end{bmatrix}$$

If $a - 5 \neq 0 \Rightarrow a \neq 5$

Then, rank of $[A] = \text{rank of} [A \vdots B] = 3$

Hence, A can take any value except 5.

Intro Exercise 3

1. If P and Q are non-singular matrices, then for the matrix M which of the following statements is correct?

(a) Rank $(PMQ) >$ Rank M
(b) Rank $(PMQ) =$ Rank M
(c) Rank $(PMQ) <$ Rank M
(d) Rank $(PMQ) =$ Rank M + Rank (PQ)

2. Solution for the system defined by the set of equations

$$4y + 3z = 8$$
$$2x - z = 2$$
$$3x + 2y = 5 \text{ is}$$

[CE GATE 2006]

(a) $x = 0, y = 1, z = \frac{4}{3}$ (b) $x = 0, y = \frac{1}{2}, z = 2$

(c) $x = 1, y = \frac{1}{2}, z = 2$ (d) Non-existent

3. Consider the system of simultaneous equations $x + 2y + z = 6$, $2x + y + 2z = 6$, $x + y + z = 5$. The system has **[ME GATE 2003]**

(a) unique solution
(b) infinite number of solutions
(c) no solution
(d) exactly two solutions

4. A is a 3×4 real matrix and $AX = b$ is an inconsistent system of equations. The highest possible rank of A is **[ME GATE 2005]**
(a) 1 (b) 2
(c) 3 (d) 4

5. The system of linear equations
$4x + 2y = 7, \quad 2x + y = 6$ has **[EC GATE 2008]**
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) exactly two distinct solutions

6. How many solutions does the following system of linear equations have?
$-x + 5y = -1,\ x - y = 2,\ x + 3y = 3$
[CS GATE 2004]
(a) Infinitely many
(b) Two distinct solutions
(c) Unique
(d) None of the above

7. Consider the following system of equations in three real variables x_1, x_2 and x_3 :
$$2x_1 - x_2 + 3x_3 = 1$$
$$3x_1 - 2x_2 + 5x_3 = 2$$
$$-x_1 - 4x_2 + x_3 = 3$$
This system of equation has **[CS GATE 2005]**
(a) no solution
(b) a unique solution
(c) more than one but a finite number of solutions
(d) an infinite number of solutions

8. If $3x + 2y + z = 0,\ x + 4y + z = 0,\ 2x + y + 4z = 0$ is a system of equations, then
(a) it is consistent
(b) only trivial solution $x = y = z = 0$ exists
(c) it can be reduced to a single equation and so a solution does not exist
(d) determinant of the matrix of coefficient is zero

9. If $x + y + z = -3,\ 3x + y - 2z = -2,\ 2x + 4y + 7z = 7$ is a system of equations, then
(a) it is inconsistent
(b) it is consistent
(c) only trivial solution $x = y = z = 0$ exists
(d) an infinite number of solutions exist

10. If $x + 2y + 2z = 1,\ 2x + y + z = -2,\ 3x + 2y + 2z = 3,\ y + z = 0$ is a system of equations, then
(a) no solution exists
(b) a unique solution exists
(c) an infinite number of solutions exist
(d) None of the above

11. If $x + 2y - 2u = 0,\ 2x - y - u = 0,\ x + 2z - u = 0,\ 4x - y + 3z - u = 0$ is a system of equations, then it is
(a) consistent with trivial solution
(b) consistent without trivial solution
(c) inconsistent with trivial solution
(d) inconsistent without trivial solution

12. If $3x + 2y + z = 0,\ x + 4y + z = 0,\ 2x + y + 4z = 0$ be a system of equations, then
(a) it is inconsistent
(b) it has only trivial solution
(c) it can be reduced to a single equation and so a solution does not exist
(d) the determinant of the matrix of coefficient is zero

13. A set of r n-vectors $x_1, x_2, \ldots, x_r$ is said to be linearly independent, if every relation of the type $k_1x_1 + k_2x_2 + \ldots + k_rx_r = 0$ implies
(a) $k_1 + k_2 + \ldots + k_r = 0$ (b) $k_1 = k_2 = \ldots = k_r = 0$
(c) $k_1 + k_2 + \ldots + k_r = 1$ (d) None of these

14. The values of k for which equations $x + y + z = 1,\ x + 2y + 4z = k,\ x + 4y + 10z = k^2$ have a solution
(a) 1 or 2 (b) 3 or 4
(c) 5 or 6 (d) Any value

15. The value of p for which system of equations $px + y = 1, x + 2y = 3$ and $2x + 3y = 5$ are consistent, is given by
(a) 1 (b) 0 (c) 2 (d) ∞

16. If the system of equations $kx + 3y - 4z = 0,\ x - ky + z = 0,\ 5x + 4y - 3z = 0$ has a non-zero solution, then k is
(a) $-2, 6$ (b) $1, -5$
(c) $-1, 5$ (d) None of these

17. Which of the following is correct solution set for $x + y + z = 6,\ x - y + z = 2,\ 2x + y - z = 1$?
(a) 1, 2, 2 (b) 2, 1, – 1
(c) 1, 2, 3 (d) 1, – 2, 3

18. For the system of equations $x + y + z = 6,\ x - y + z = 2$ and $2x + y - z = 1$, which is correct?
(a) Consistent
(b) Consistent with unique solution
(c) No solution
(d) None of the above

19. The equations $kx + y + z = 0,\ -x + ky + z = 0,\ -x - y + kz = 0$ will have a non-zero solution, when real k is
(a) 3 (b) zero (c) 1 (d) $\sqrt{3}$

20. The system of equations $kx + 2y - z = 1,\ (k-1)y - 2z = 2,\ (k+2)z = 3$ will have a unique solution if k is
(a) zero (b) 1 (c) – 1 (d) – 2

Answers with Solutions

1. (d) **2.** (c)

3. (c) The given system of equations can be written as

$$AX = B$$

where, $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, b = \begin{bmatrix} 6 \\ 6 \\ 5 \end{bmatrix}$

The augmented matrix is

$$[A : b] = \begin{bmatrix} 1 & 2 & 1 & \vdots & 6 \\ 2 & 1 & 2 & \vdots & 6 \\ 1 & 1 & 1 & \vdots & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & \vdots & 6 \\ 0 & 3 & 0 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 1 & 0 & \vdots & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & \vdots & 6 \\ 0 & 1 & 0 & \vdots & 2 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

$\therefore \quad \rho(A) = 2$ and $\rho(A : b) = 3$

$\Rightarrow \quad \rho(A) \neq \rho[A : B] \Rightarrow$ No solution

4. (b)

5. (b) Given, system of equations can be written as

$$AX = B$$

where, $A = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 4 & 2 & \vdots & 7 \\ 2 & 1 & \vdots & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & \vdots & \frac{7}{4} \\ 2 & 1 & \vdots & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1/2 & \vdots & 7/4 \\ 0 & 0 & \vdots & -5/6 \end{bmatrix}$$

$\therefore \quad \rho(A) = 1, \rho[A : B] = 2$

$\therefore \quad \rho(A) \neq \rho[A : B]$

$\Rightarrow$ No solution

6. (c) Given system of equations can be written as

$$AX = B$$

where, $A = \begin{bmatrix} -1 & 5 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$

The augmented matrix is

$$[A : B] = \begin{bmatrix} -1 & 5 & \vdots & -1 \\ 1 & -1 & \vdots & 2 \\ 1 & 3 & \vdots & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & \vdots & 1 \\ 0 & -4 & \vdots & -1 \\ 0 & -8 & \vdots & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -5 & \vdots & 1 \\ 0 & 1 & \vdots & 1/4 \\ 0 & 0 & \vdots & 0 \end{bmatrix}$$

$\therefore \rho(A) = \rho[A : B] = 2 =$ Number of variables.

$\therefore$ Unique solution.

7. (b) Given system of equations can be written as

$$AX = B$$

where, $A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & -2 & 5 \\ -1 & -4 & 1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

The augmented matrix is

$$[A : B] = \begin{bmatrix} 2 & -1 & 3 & \vdots & 1 \\ 3 & -2 & 5 & \vdots & 2 \\ -1 & -4 & 1 & \vdots & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 & \vdots & 1/2 \\ 0 & 1/2 & -1/2 & \vdots & -1/2 \\ 0 & -9/2 & 5/2 & \vdots & 7/2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1/2 & 3/2 & \vdots & 1/2 \\ 0 & 1 & -1 & \vdots & -1 \\ 0 & 0 & -2 & \vdots & -1 \end{bmatrix}$$

$\therefore \rho(A) = \rho(A : B] = 3 =$ Number of variables

$\therefore$ Unique solution.

8. (b) **9.** (a) **10.** (b) **11.** (a) **12.** (b)
13. (b) **14.** (a) **15.** (b) **16.** (d) **17.** (c)
18. (b) **19.** (b) **20.** (c)

Eigen Values and Eigen Vectors

Vectors

Any ordered n-tuple of numbers is called an n-vector. By an ordered n-tuple, we mean a set consisting of n numbers in which the place of each number is fixed. If $x_1, x_2, \dots, x_n$ be any n numbers, then the ordered n-tuple $X = (x_1, x_2, \dots, x_n)$ is called an n-vector. Thus, the coordinates of a point in space can be represented by a 3-vector (x, y, z). Similarly, $(1, 0, 2, -1)$ and $(2, 7, 5, -3)$ are 4-vectors. The n numbers $x_1, x_2, \dots, x_n$ are called the components of the n-vector $X = (x_1, x_2, \dots, x_n)$. A vector may be written either as a row vector or as a column vector. If A is a matrix of order $m \times n$, then each row of A will be an n-vector and each column of A will be an m-vector. A vector whose components are all zero is called a zero vector and is denoted by O. Thus $O = (0, 0, 0, \dots, 0)$.

Let $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ be two vectors.

Then $X = Y$ if and only if their corresponding components are equal.

i.e., if $x_i = y_i$ for $i = 1, 2, \dots, n$

$$X + Y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

If k be a scalar, then $kX = (kx_1, kx_2, \dots, kx_n)$

n, n-vectors $e_1 = [1, 0, 0, \dots, 0]$, $e_2 = [0, 1, 0, 0, \dots, 0]$, …, $e_n = [0, 0, \dots, 1]$ are called fundamental unit vectors or elementary vectors.

Linear Dependence and Linear Independence of Vectors

A set of r n-vectors $X_1, X_2, \dots, X_r$ is said to be linearly dependent if there exist r scalars (numbers) $k_1, k_2, \dots, k_r$ not all zero, such that

$$k_1 X_1 + k_2 X_2 + \dots + k_r X_r = O$$

A set of r n-vectors $X_1, X_2, \dots, X_r$ is said to be linearly independent if every relation of the type

$$k_1 X_1 + k_2 X_2 + \dots + k_r X_r = O \text{ implies}$$
$$k_1 = k_2 = \dots = k_r = 0$$

To test the linear dependence of r given vectors, write them as row vectors. Add suitable multiples of one vector to the others so that the resulting $(r-1)$ vectors have their first component zero. Choose anyone of these $(r-1)$ vectors and add its multiples to the others so that the resulting $(r-2)$ vectors have their second component zero. In this way continue, reducing the successive components to zero. If the final reduction gives a vector all of whose components are zero, then the original vectors are linearly dependent. However, if the final reduction gives a vector all of whose components are not zero, then the original vectors are linearly independent.

- If a set of vectors is linearly dependent, then atleast one member of the set can be expressed as a linear combination of the remaining vectors.
- The m, n-dimensional vectors $X_1, X_2, X_3, \dots, X_m$ are linearly dependent if the rank of the matrix $[X_1, X_2, X_3, \dots, X_m]$ with the given vectors as columns is less than m.

Procedure to Test Linear Dependence of Vectors

Step 1. Construct coefficient matrix A with elements of given vectors as columns.

Step 2. Find the rank of A.

Step 3. If $\rho(A) =$ number of vectors then given set of vectors is linearly independent and if $\rho(A) <$ number of vectors then given set of vectors is linearly dependent.

Any Non-empty Subset of a Linearly Independent Set of Vectors is Linearly Independent

Let $A_1 = \{X_1, X_2, \dots, X_n\}$

and $A_2 = \{X_1, X_2, \dots, X_r\}, r < n$

From above, we see that A_2 is a subset of A_1.

So, let $a_1 X_1 + a_2 X_2 + a_3 X_3 + \dots + a_r X_r = 0$ …(i)

or $a_1 X_1 + a_2 X_2 + a_3 X_3 + \dots + a_r X_r + 0X_{r+1} + 0X_{r+2} + \dots + 0 \cdot X_n = 0$ …(ii)

But A_1 is linearly independent so from Eq. (ii),

$$a_1 = a_2 = a_3 = a_4 = \dots = a_r = 0$$

On putting in Eq. (i), we get that the set A_2 is also linearly independent.

Any Superset of a Linearly Dependent Set of Vectors is Linearly Dependent

Let the set $X = \{X_1, X_2, X_3, \dots, X_m\}$ be linearly dependent set so $\exists$ scalars $a_1, a_2, a_3, \dots, a_r$ not all zero but some are zero such that

$$a_1X_1 + a_2X_2 + a_3X_3 + \ldots + a_mX_m = 0 \quad \ldots\text{(i)}$$

or $$a_1X_1 + a_2X_2 + a_3X_3 + \ldots + a_mX_m + 0 \cdot X_{m+1} + 0X_{m+2} + \ldots + 0X_n = 0 \quad \ldots\text{(ii)}$$

where, all a's are not zero. So, by definition of linearly dependent given set

$X = [X_1, X_2, \ldots, X_n]$ is linearly dependent.

Eigen Values and Eigen Vectors

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n over the field $F = (C$ or R or $\theta)$. Then $\lambda \in F$ is called an eigen value of A if $\exists\, X (\neq 0) \in F^n$ such that

$$AX = \lambda X$$

Now, $AX = \lambda X, X \neq 0$

$\Rightarrow$ $(A - \lambda I)\, X = 0, X \neq 0$

$\Rightarrow$ $BX = 0, X \neq 0, B = A - \lambda I$

$\Rightarrow$ For non-trivial solution $|B| = 0$ *i.e.*, B is singular

i.e., $\rho(B) < n$

$\Rightarrow$ $|A - \lambda I| = 0$

This is known as characteristic equation of matrix A.

The matrix

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots\ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ a_{n1} & a_{n2} & \ldots\ldots & a_{nn} - \lambda \end{bmatrix}$$

is known as characteristic matrix and $f(\lambda) = |A - \lambda I|$ is known as characteristic polynomial of A in λ. This is a monic polynomial.

Thus, the roots of characteristic equation say $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of matrix A. Eigen values are also called eigen roots or latent roots or characteristic values or characteristic roots of the matrix A.

The set of the eigen values of the matrix A is called the spectrum of the matrix A and the largest eigen value is called the spectral radius of A.

If λ is a characteristic root of an $n \times n$ matrix A, then the non-zero solution

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

of the equation $AX = \lambda X$, *i.e.*, $(A - \lambda I)\, X = 0$ is called the characteristic vector or eigen vector of the matrix A.

It is to be noted that if X is a non-zero solution then kX, where k is a constant, is also a solution of the homogeneous system. Hence, an eigen vector is unique only upto a constant multiple.

The problem of finding eigen values and the corresponding eigen vector of a square matrix A is called an eigen value problem.

Characteristics of Eigen Values

(i) Any square matrix A and its tranpose both have the same eigen values.

(ii) The trace of the matrix equals the sum of the eigen values of a matrix.

(iii) The determinant of the matrix A equals to the product of the eigen values of A.

Proof Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a square matrix of order 3.

Now, $|A - \lambda I| = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{bmatrix}$

$$= \lambda^3 + \lambda^2 (a_{11} + a_{22} + a_{33})$$
$$+ \lambda\, (-a_{11}a_{22} - a_{11}a_{33} + a_{12}a_{21} - a_{22}a_{33} + a_{23}a_{32} + a_{33}a_{31})$$
$$- [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}\,(a_{21}a_{32} - a_{31}a_{22})] \quad \ldots\text{(i)}$$

Let λ_1, λ_2 and λ_3 are the roots of the Eq. (i), then

sum of the roots $= \lambda_1 + \lambda_2 + \lambda_3 = a_{11} + a_{22} + a_{33}$

= sum of the diagonal elements.

i.e., trace of matrix equals the sum of the eigen values of matrix A.

Product of the roots $= \lambda_1\lambda_2\lambda_3$

$$= [a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})]$$

= Determinant A.

Hence, determinant of matrix A equals the product of the eigen value of A.

(iv) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the n-eigen values of square matrix A, then

(a) $\lambda_r + k$ is eigen value of $A + kI$, $k \in R$

(b) Eigen values of kA are $k\lambda_1, k\lambda_2, \ldots, k\lambda_n$.

(c) If λ is an eigen value of a matrix A, then $\frac{1}{\lambda}$ is the eigen value of A^{-1}, provided A is on-singular matrix.

If X be the eigen vector corresponding to λ then,

$$AX = \lambda X$$

[Multiplying by A^{-1} to both side in preface]

or $A^{-1}AX = A^{-1}\lambda X$

or $IX = \lambda(A^{-1}X)$ $[\because AA^{-1} = A^{-1}A = I]$

$\Rightarrow$ $X = \lambda(A^{-1}X)$ $[\because IA = AI = A]$

or $A^{-1}X = \frac{1}{\lambda}X$

$\Rightarrow$ This is same as Eq. (i).

Hence, $\frac{1}{\lambda}$ is an eigen value of the inverse matrix A^{-1}.

Corollary If $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are n-eigen values of matrix A, then eigen values of A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$, provided $|A| \neq 0$.

(d) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A, then A^m has the eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$ where m is a positive integer.

Theorem 1. Any system of characteristic vectors $X_1, X_2, \ldots, X_k$ respectively corresponding to a system of distinct characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ of any square matrix A are linearly independent.

Proof By definition, we have

$$AX_1 = \lambda_1 X_1,\ AX_2 = \lambda_2 X_2, \ldots, AX_k = \lambda_k X_k \quad \ldots\text{(i)}$$

$$\Rightarrow \quad (A - \lambda_1 I)X_1 = 0, \ldots, (A - \lambda_k I)X_k = 0 \quad \ldots\text{(ii)}$$

Now, $X_1, X_2, \ldots, X_k$ will be linearly independent when

$$c_1X_1 + c_2X_2 + \ldots + c_kX_k = 0 \quad \ldots\text{(iii)}$$

For each $c_i = 0\ (i = 1, 2, \ldots, k)$

Now, let $P_i = (A - \lambda_1 I)(A - \lambda_2 I)\ldots(A - \lambda_k I)$, where the factor $A - \lambda_i I$ is missing.

Then, we have $P_1 = (A - \lambda_2 I)\ldots(A - \lambda_k I)$, here $A - \lambda_1 I$ is missing.

$$\begin{aligned}
\Rightarrow \quad P_1X_1 &= [(A - \lambda_2 I)\ldots(A - \lambda_k I)]X_1 \\
&= (A - \lambda_2 I)\ldots(AX_1 - \lambda_k X_1) \\
&= (A - \lambda_2 I)\ldots(\lambda_1 X_1 - \lambda_k X_1) \\
&= (\lambda_1 - \lambda_k)\ldots(A - \lambda_2 I)X_1 \\
&= (\lambda_1 - \lambda_k)(\lambda_1 - \lambda_{k-1})\ldots(\lambda_1 - \lambda_2)X_1, \text{ etc.}
\end{aligned}$$

While $P_1X_2 = 0, \ldots, P_1X_k = 0$, then pre-multiplying Eq. (iii) by P_1 and making use of the above relations, we get

$$P_1(C_1X_1 + C_2X_2 + C_3X_3 + \ldots + C_kX_k) = 0$$

$$\Rightarrow \quad P_1(C_1X_1) = 0$$

$$\Rightarrow \quad C_1P_1X_1 = 0$$

$$\Rightarrow \quad C_1(\lambda_1 - \lambda_k)(\lambda_1 - \lambda_{k-1})\ldots(\lambda_1 - \lambda_2)X_1 = 0$$

But λ's are distinct and also $X_1 \neq 0$, so C_1 must be zero. Similarly, if we multiply Eq. (iii) by P_2, we get $C_2 = 0$ and in general if we multiply Eq. (iii) by P_i we would get $C_i = 0$.

Continuing in this way, we can say that the vectors $X_1, \ldots, X_k$ are linearly independent.

Theorem 2. The matrix A and its transpose A' have the same characteristic root.

Proof Characteristic equation of matrix A is given by

$$\det(A - \lambda I) = |A - \lambda I| = 0 \quad \ldots\text{(i)}$$

Also characteristic equation of matrix A' is given by

$$\det(A' - \lambda I) = |A' - \lambda I| = 0 \quad \ldots\text{(ii)}$$

Obviously, Eqs. (i) and (ii) are same, since $|A| = |A'|$.

Theorem 3. If A and P be square matrices of the same type and if P be invertible, show that the matrices A and $P^{-1}AP$ have the same characteristic root.

Proof Let $B = P^{-1}AP$, then we will show that characteristic equations for both A and B are the same and hence they have the same characteristic root.

Now, $B - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P$

$$[\because P^{-1}\lambda IP = \lambda I]$$

$$\begin{aligned}
\therefore \quad |B - \lambda I| &= |P^{-1}(A - \lambda I)P| = |P^{-1}||A - \lambda I||P| \\
&= |A - \lambda I||P^{-1}||P| = |A - \lambda I||P^{-1}P| \\
&= |A - \lambda I||I| = |A - \lambda I| \quad [\because |I| = 1]
\end{aligned}$$

Hence, the matrices A and B have the same characteristic equation, *i.e.,* the same characteristic roots.

Theorem 4. If A and B be n-rowed square matrices and if A be invertible, show that the matrices $A^{-1}B$ and BA^{-1} have the same characteristic root.

Proof We know

$$A^{-1}B = A^{-1}B(A^{-1}A) = A^{-1}(BA^{-1})A. \quad [\because A^{-1}A = I]$$

But matrices BA^{-1} and $A^{-1}(BA^{-1})A$ have same characteristic root (Theorem 3).

So, matrices BA^{-1} and $A^{-1}B$ also have same characteristic root.

Theorem 5. Show that O is a characteristic root of a matrix if and only if the matrix is singular.

Proof Characteristic equation of matrix A is

$$\det(A - \lambda I) = |A - \lambda I| = 0$$

When $\lambda = 0$, then $|A| = 0$

i.e., Matrix A is singular and when the matrix is singular, we have

$$|A - \lambda I| = 0 \Rightarrow |A| - \lambda |I| = 0$$

$$\Rightarrow \quad 0 - \lambda \cdot 1 = 0 \Rightarrow \lambda = 0$$

Theorem 6. If A and B be two square invertible matrices, then AB and BA have the same characteristic root.

Proof We know,

$$AB = IAB = B^{-1} B(AB) = B^{-1}(BA)B$$

But matrices BA and $B^{-1}(BA)B$ have same characteristic root.

Hence, matrices BA and AB have the same characteristic root.

Theorem 7. λ is a characteristic root of a matrix A if and only if there exists a non-zero vector X such that $AX = \lambda X$.

Proof Let A be a square matrix of order n, and if λ is a characteristic root of the matrix A, then $|A - \lambda I| = 0$, *i.e.,* the characteristic matrix $A - \lambda I$ is singular.

Hence, there exists a non-zero solution of the equation

$$(A - \lambda I)X = 0$$

or

$$AX = \lambda X$$

i.e., there exists a non-zero vector X such that, $AX = \lambda X$.

Converse Let there exists a non-zero vector X, such that

$$AX = \lambda X$$

i.e., there exists a non-zero solution of the equation $(A - \lambda I)X = 0$.

The coefficient matrix $A - \lambda I$ must be singular.

Hence, $|A - \lambda I| = 0$

$\Rightarrow \lambda$ is a characteristic root of the matrix A.

Theorem 8. The characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Proof Let the triangular matrix be given as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} \end{bmatrix}$$

So, its characteristic equation is

$$|A - \lambda I| = 0$$

or

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & a_{22} - \lambda & a_{23} & a_{24} & a_{25} \\ 0 & 0 & a_{33} - \lambda & a_{34} & a_{35} \\ 0 & 0 & 0 & a_{44} - \lambda & a_{45} \\ 0 & 0 & 0 & 0 & a_{55} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)(a_{44} - \lambda)(a_{55} - \lambda) = 0$$

or

$$\lambda = a_{11}, a_{22}, a_{33}, a_{44}, a_{55}$$

The above elements are diagonal elements of given matrix A.

Theorem 9. All the characteristic roots of a Hermitian matrix are real.

Proof Let A be Hermitian matrix and λ be its eigen value. If X be eigen vector corresponding to eigen value λ, then

$$AX = \lambda X \qquad \text{...(i)}$$

Premultiplying both sides of Eq. (i) by X^θ, we have

$$X^\theta AX = X^\theta \lambda X = \lambda X^\theta X \qquad \text{...(ii)}$$

where, X^θ is the transposed conjugate of X.

Taking transpose-conjugate of both sides of Eq. (ii), we have

$$(X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$$

$$(AX)^\theta (X^\theta)^\theta = (X^\theta X)^\theta \bar{\lambda} \qquad (\text{since } \lambda^\theta = \bar{\lambda})$$

Since, A is a Hermitian matrix hence $A^\theta = A$ and $(X^\theta)^\theta = X$, therefore

$$X^\theta A^\theta X = X^\theta X \bar{\lambda}$$

or

$$X^\theta AX = \bar{\lambda} X^\theta X \qquad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get

$$\lambda X^\theta X = \bar{\lambda} X^\theta X$$

or

$$(\lambda - \bar{\lambda}) X^\theta X = 0$$

But X is not a zero vector, hence $X^\theta X \neq 0$

Hence,

$$\lambda - \bar{\lambda} = 0$$

$\Rightarrow \lambda = \bar{\lambda}$, which is only possible when λ is real.

Corollary 1. Characteristic root of a Skew-Hermitian matrix is either zero or a pure imaginary number.

Proof Let A be a Skew-Hermitian matrix.

$\Rightarrow$ iA is Hermitian.

Now, if λ is a characteristic root of A

Then,

$$AX = \lambda X$$

$$\Rightarrow \quad (iA)X = (i\lambda)X$$

$\Rightarrow i\lambda$ is the characteristic root of (iA)

$\Rightarrow i\lambda$ should be real, which is possible only when λ is either zero or a pure imaginary number.

Corollary 2. Characteristic roots of a real symmetric matrix are all real.

Proof For a given symmetric matrix,

$$A' = A$$

$$\Rightarrow \quad (\bar{A}') = A$$

or $$A^{\theta} = A$$

$\Rightarrow A$ is called Hermitian matrix.

Hence, all characteristic roots are real.

Theorem 10. The modulus of each characteristic root of a unitary matrix is unity.

Proof Let A be a unitary matrix then

$$A^{\theta} \cdot A = A \cdot A^{\theta} = I$$

Now, since λ is a characteristic root of A,

So, $$AX = \lambda X$$

Taking transpose conjugate to both sides,

$$(AX)^{\theta} = (\lambda X)^{\theta}$$

$$\Rightarrow \quad X^{\theta} A^{\theta} = \bar{\lambda} X^{\theta}$$

Now, multiplying above by $AX = \lambda X$

$$X^{\theta} A^{\theta} AX = \bar{\lambda} X^{\theta} \lambda X$$

$$\Rightarrow \quad X^{\theta} A^{\theta} AX = \bar{\lambda}\lambda X^{\theta} X$$

$$\Rightarrow \quad X^{\theta} IX = \bar{\lambda}\lambda X^{\theta} X \qquad [\because A^{\theta} A = I]$$

$$\Rightarrow \quad (1 - \lambda\bar{\lambda}) X^{\theta} X = 0$$

But $\because \quad X \neq 0 \quad \Rightarrow \quad X^{\theta} \neq 0$

So, $$1 - \lambda\bar{\lambda} = 0$$

$$\Rightarrow \quad \lambda\bar{\lambda} = 1$$

$$\Rightarrow \quad |\lambda|^2 = 1$$

Hence, the modulus of characteristic roots of a unitary matrix is unity.

- Any two characteristic vector corresponding to two distinct characteristic roots of a Hermitian matrix are orthogonal.
- Any two characteristic vector corresponding to two distinct characteristic roots of a real symmetric matrix are also orthogonal.
- $P(\lambda)$ is an eigen value of $P(A)$, where

$$P(A) = a_0 I + a_1 A + a_2 A^2 + \ldots + a_n A^n$$

- If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then $\frac{|A|}{\lambda_1}, \frac{|A|}{\lambda_2}, \frac{|A|}{\lambda_3}, \ldots, \frac{|A|}{\lambda_n}$ are the eigen values of Adj (A).
- The maximum number of eigen values are equal to the size of matrix, *i.e.,* equal to the number of rows.
- In the diagonal matrix and triangular matrix (upper and lower triangular matrix) the diagonal elements are only the eigen values of the matrix.

Example 1. Consider the system of equations $A_{(n \times n)} x_{(n \times t)} = \lambda(n \times l)$ where, λ is a scalar. Let (λ_i, x_i) be an eigen-pair of an eigen value and its corresponding eigen vector for real matrix A. Let I be a $(n \times n)$ unit matrix. Which one of the following statements is not correct? **[CE GATE 2005]**

(a) For a homogeneous $n \times n$ system of linear equations, $(A - \lambda I)x = 0$ having a non-trivial solution, the rank of $(A - \lambda I)$ is less than n.

(b) For matrix A^m, m being a positive integer, (λ_i^m, x_i^m) will be the eigen pair for all i.

(c) If $A^T = A^{-1}$, then $|\lambda_i| = 1$ for all i.

(d) If $A^T = A$, then λ_i is real for all i.

Sol. (b) For matrix A^m, m being a positive integer $(\lambda_i^m \; X_i^m)$ will be the eigen pair for all i.
Although λ_i^m will be the corresponding eigen values of A^m, X_i^m will not be corresponding eigen vectors.

Example 2. For a given matrix

$$A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$$

one of the eigen values is 3. The other two eigen values are **[CE GATE 2006]**

(a) 2, – 5 (b) 3, – 5
(c) 2, 5 (d) 3, 5

Sol. (b) $\because \quad \Sigma\lambda_i = \text{Trace}(A)$

$$\lambda_1 + \lambda_2 + \lambda_3 = \text{Trace}(A) = 2 + (-1) + 0 = 1$$

Now, $$\lambda_1 = 3$$

$\therefore$ $$3 + \lambda_2 + \lambda_3 = 1$$

$\Rightarrow$ $$\lambda_2 + \lambda_3 = -2$$

Only choice (b) satisfies this condition.

Example 3. Eigen values of a matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5 and 1. What is the eigen values of the matrix $S^2 = SS$? **[ME GATE 2006]**

(a) 1 and 25 (b) 6 and 4
(c) 5 and 1 (d) 2 and 10

Sol. (a) ∵ If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of A, then the eigen values of A^m are $\lambda_1^m, \lambda_2^m, \ldots, \lambda_n^m$.

∵ S has eigen values 1 and 5 so S^2 has eigen values 1^2 and 5^2.

$\Rightarrow$ 1 and 25

Example 4. The eigen values of the matrix

$$\begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \text{are}$$

[CE GATE 2004]

(a) 1 and 4
(b) – 1 and 2
(c) 0 and 5
(d) Cannot be determined

Sol. (c) ∵ We have,

$$A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

So, its characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (4-\lambda)(1-\lambda) - 4 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \quad \lambda = 0, 5$$

Hence, $\lambda = 0$ and 5 are the eigen values.

Example 5. For the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ the eigen values are

[ME GATE 2003]

(a) 3 and – 3 (b) – 3 and – 5
(c) 3 and 5 (d) 5 and 0

Sol. (c) ∵ We have, $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$

So, its characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (4-\lambda)(4-\lambda) - 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 8\lambda + 15 = 0$$

$$\Rightarrow \quad (\lambda-3)(\lambda-5) = 0$$

$$\Rightarrow \quad \lambda = 3 \text{ and } 5$$

Example 6. For the matrix $p = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to – 2. Which of the following is an eigen vector?

[EE GATE 2005]

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

Sol. (d) Since matrix is triangular, the eigen values are the diagonal elements themselves namely $\lambda = 3, -2$ and 1. Corresponding to eigen value, $\lambda = -2$ let us find the eigen vector

$$[A - \lambda I]\, X = 0$$

$$\begin{bmatrix} 3-\lambda & -2 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Putting $\lambda = -2$ in above equation, we get

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

which gives the equations,

$$5x_1 - 2x_2 + 2x_3 = 0 \quad \ldots\text{(i)}$$

$$x_3 = 0 \quad \ldots\text{(ii)}$$

$$3x_3 = 0 \quad \ldots\text{(iii)}$$

Since, Eqs. (ii) and (iii) are same, we have

$$5x_1 - 2x_2 + 2x_3 = 0$$

$$x_3 = 0$$

Putting $x_2 = k$ in Eq. (i), we get

and $5x_1 - 2k + 2 \times 0 = 0$

$$\Rightarrow \quad x_1 = \frac{2}{5}k$$

∴ Eigen vectors are of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2/5\,k \\ k \\ 0 \end{bmatrix}$$

i.e., $x_1 : x_2 : x_3 = \frac{2}{5}k : k : 0 = \frac{2}{5} : 1 : 0 = 2 : 5 : 0$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \text{ is an eigen vector of matrix } p.$$

Example 7. Which one of the following is an eigen vector of the matrix

$$\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}?$$

[ME GATE 2008]

(a) $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}$

Sol. (a) $\because$ We have, $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$

So, the characteristic equation is $|A - \lambda I| = 0$

or $$\begin{bmatrix} 5-\lambda & 0 & 0 & 0 \\ 0 & 5-\lambda & 5 & 0 \\ 0 & 0 & 2-\lambda & 1 \\ 0 & 0 & 3 & 1-\lambda \end{bmatrix} = 0$$

$\Rightarrow \quad (5-\lambda)(5-\lambda)[(2-\lambda)(1-\lambda) - 3] = 0$

$\Rightarrow \quad (5-\lambda)^2 (\lambda^2 - 3\lambda - 1) = 0$

$\Rightarrow \quad \lambda = 5, 5 \quad \text{and} \quad \lambda = \dfrac{3 \pm \sqrt{13}}{2}$

For eigen vector, putting $\lambda = 5$ in $[A - \lambda I]\, X = 0$

$$\begin{bmatrix} 5-5 & 0 & 0 & 0 \\ 0 & 5-5 & 5 & 0 \\ 0 & 0 & 2-5 & 1 \\ 0 & 0 & 3 & 1-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 5x_3 = 0 \quad \ldots\text{(i)}$

$-3x_3 + x_4 = 0 \quad \ldots\text{(ii)}$

$3x_3 - 4x_4 = 0 \quad \ldots\text{(iii)}$

On solving Eqs. (i), (ii) and (iii), we get

$x_3 = 0,\ x_4 = 0$ and x_1, x_2 may be anything.

The eigen vector corresponding to $\lambda = 5$, may be written as

$$\dot{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \\ 0 \end{bmatrix}$$

where k_1 and k_2 may be any real number. Since, choice (a) is the only matrix in this form with both x_3 and $x_4 = 0$, so it is the correct answer.

Example 8. The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

[ME GATE 2007]

(a) 0 (b) 1 (c) 2 (d) infinite

Sol. (b) $\because$ We have, $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

So, characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^2 - 0 = 0$

$\Rightarrow \quad \lambda = 2, 2$

Now, eigen vector for $\lambda = 2$

Put $\quad [A - \lambda I]\, X = 0$

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

On solving, $x_2 = 0$ and $0 = 0$

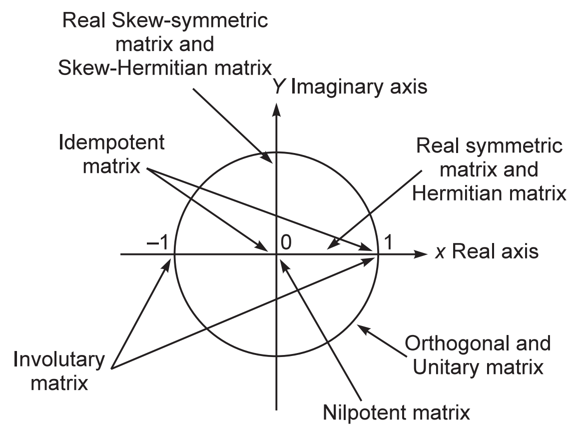
Hence, $x_2 = 0$ and x_1 may be anything.

Hence, let $x_1 = k$

Then, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k \\ 0 \end{bmatrix}$

So, there is only one linearly independent eigen vector for the given problem which can be written as $\begin{bmatrix} k \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Spectrum of Eigen Values of Matrices



Algebraic Multiplicity and Geometric Multiplicity

Let $A = [a_{ij}]_{n \times n}$ be a square matrix of order n whose characteristic equation is

$$\text{charc}\,(x) = 0$$

If $x = \lambda$ is a root of ch $(x) = 0$ such that

$$\text{ch}\,(\lambda) = \text{ch}'\,(\lambda) = \ldots = \text{ch}^{r-1}\,(\lambda) = 0$$

But $\quad \text{ch}^{r}\,(\lambda) \neq 0$

Then, r is called algebraic multiplicity of λ.

In other words, the multiplicity of an eigen value is called algebraic multiplicity of that eigen value *i.e.,* if

$$\text{ch}\,(x) = (x - \lambda)^r\, \phi(x)$$

Then, r is algebraic multiplicity of eigen value λ.

Now, construct the matrices $B_r = A - \lambda_r I$ (distinct), where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are eigen values of matrix A. Thus, we get system of homogeneous equations.

$$B_r X = 0 \,;\, r = 1 \text{ to } k.$$

Now, W_r = solution set of $B_r X = 0$

$$= \{X : B_r X = 0\}$$

W_r is called eigen space corresponding to eigen value λ_r. Moreover, W_r is null space of B_r. Thus,

$$\text{Dim } B_r = \text{Nullity of } B_r$$

$$= n - \text{Rank } B_r = m_r$$

This m_r is called the geometric multiplicity of λ_r, which is equal to the number of linearly independent eigen vectors of matrix A.

Thus, geometric multiplicity of λ_r

= Number of linearly independent eigen vector of A

$= n - \rho(B_r) = n - \rho(A - \lambda_r I) = m_r$

Clearly, Algebraic Multiplicity (AM) of λ_r is greater than or equal to the geometric multiplicity of λ_r.

Example 9. For the matrix

$$A = \begin{bmatrix} 1 & 2010 & 2011 \\ 0 & -1 & 2010 \\ 0 & 0 & 1 \end{bmatrix}$$

find the algebraic multiplicity and geometric multiplicity of the eigen values.

Sol. $\because$ ch $(x) = (x-1)^2 (x+1)$

$\therefore$ The eigen values of the given matrix are 1, 1, and -1.

$\therefore$ AM of $\lambda = 1$ is 2

and AM of $\lambda = -1$ is 1

Now, constructing

$$B_1 = A - I = \begin{bmatrix} 0 & 2012 & 2011 \\ 0 & -2 & 2010 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_{-1} = A + I = \begin{bmatrix} 2 & 2012 & 2011 \\ 0 & 0 & 2010 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore$ $\rho(B_1) = 2$

$\rho(B_{-1}) = 3$

$\therefore$ GM of $\lambda = 1$ is $3 - 2 = 1$

and GM of $\lambda = -1$ is $3 - 3 = 0$.

Thus, the number of eigen vector corresponding to $\lambda = 1$ is 1 while corresponding to $\lambda = -1$ is zero.

Minimal Polynomial

The smallest degree monic polynomial $m_A(x)$ is called the minimal polynomial for the matrix A, if

$$m_A(A) = 0$$

Properties of Minimal Polynomial

(i) For every matrix, minimal polynomial is unique.

(ii) For any matrix A, minimal polynomial divides the characteristic polynomial of A *i.e.,*

$$\deg m_A(x) \le \deg \text{ch}(x).$$

(iii) The deg $m_A(x) = 1$ iff A is scalar matrix.

Thus, if A is not scalar. Then

$$\deg m_A(n) \ge 2$$

(iv) If $n - \lambda$ divides $m_A(x)$ or $m_A(\lambda) = 0$, then λ is an eigen value of A *i.e.,* every root of $m_A(x)$ is an eigen value of A which does not give algebraic multiplicity of eigen value.

(v) The order of λ as zero of $m_A(x)$ may not be algebraic multiplicity of eigen value.

(vi) If $A = [a_{ij}]_{n \times n}$ is a square matrix of order n, then ch (x) divides $[m_A(x)]^n$ *i.e.,* every irreducible factor of ch (x) is also a factor of $m_A(x)$.

Hence, all the eigen values of A without algebraic multiplicity can be root of $m_A(x) = 0$. Hence, if

$$\text{ch}(x) = \pi(x - \lambda_i)^{n_i}, \Sigma n_i = n$$

Then, $m_A(x) = \pi(x - \lambda_i)^{m_i}, \Sigma m_i \le n$

Example 10. Find the minimal polynomial for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 1 & 1 & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1 \end{bmatrix}_{10 \times 10}$$

Sol. We have,

$$A^2 = \begin{bmatrix} 10 & 10 & & 10 \\ 10 & 10 & \ldots & 10 \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 10 & 10 & \ldots & 10 \end{bmatrix}_{10 \times 10} = 10A$$

$\Rightarrow$ $A^2 - 10A = 0$

$\Rightarrow$ $m_A(x) = 0$

Example 11. If A is non-scalar, non-identity idempotent matrix of order $n \ge 2$. Then minimal polynomial $m_A(x)$ is

(a) $x(x-1)$

(b) $x(x+1)$

(c) $x(1-x)$

(d) $x^2(1+x)$

Sol. (a) We have,

$$A^2 = A$$

$$\Rightarrow \quad A^2 - A = 0$$

$$\Rightarrow \quad m_A(x) = x^2 - x = x(x-1)$$

Example 12. If A is non-scalar, non-identity involutory matrix, then minimal polynomial $m_A(x)$ is

(a) $x(x-1)$ (b) $(x-1)(x+1)$
(c) $x(1-x)$ (d) $x+1$

Sol. (b) We have,

$$A^2 = I$$

$$\Rightarrow \quad A^2 - I = 0$$

$$\Rightarrow \quad m_A(x) = x^2 - 1 = (x-1)(x+1)$$

Example 13. All the four entries of the 2×2 matrix $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ are non-zero, and one of its eigen values is zero. Which of the following statements is true?

[EC GATE 2008]

(a) $p_{11}p_{22} - p_{12}p_{21} = 1$ (b) $p_{11}p_{22} - p_{12}p_{21} = -1$
(c) $p_{11}p_{22} - p_{12}p_{21} = 0$ (d) $p_{11}p_{22} + p_{12}p_{21} = 0$

Sol. (c)$\because$ All entries of given matrix are non-zero, then $|P| = 0$.

or $$\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = 0$$

$$\Rightarrow \quad p_{11}p_{22} - p_{12}p_{21} = 0$$

Example 14. One of the eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is

[ME GATE 2010]

(a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Sol. (a) $\because \quad A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

So, the characteristic equation of A is

$$|A - \lambda I| = 0$$

or $$\begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \quad \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda = 1, 4$$

The eigen value problem is $[A - \lambda I]\hat{X} = 0$

$$\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \ldots(i)$$

Putting $\lambda = 1$ in Eq. (i),

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 = 0 \quad \ldots(ii)$$

$$x_1 + 2x_2 = 0 \quad \ldots(iii)$$

Solution is $x_2 = k$, $x_1 = -2k$

$$\dot{X}_1 = \begin{bmatrix} -2k \\ k \end{bmatrix}, \; i.e., \; x_1 : x_2 = -2 : 1$$

Since, choice (a) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is in same ratio of x_1 to x_2.

Hence, (a) is the only answer.

Example 15. The matrix $\begin{bmatrix} 1 & 2 & 4 \\ 3 & 0 & 6 \\ 1 & 1 & p \end{bmatrix}$ has one eigen value equal to 3. The sum of the other two eigen values is

[ME GATE 2008]

(a) p (b) $p-1$
(c) $p-2$ (d) $p-3$

Sol. (c) $\because$ Sum of the eigen values of matrix
= Sum of diagonal values present in the matrix

$$\therefore \quad 1 + 0 + p = 3 + \lambda_2 + \lambda_3$$

$$\Rightarrow \quad p + 1 = 3 + \lambda_2 + \lambda_3$$

$$\Rightarrow \quad \lambda_2 + \lambda_3 = p + 1 - 3 = p - 2$$

Example 16. The trace and determinant of a 2×2 matrix are known to be -2 and -35 respectively. Its eigen values are

[EE GATE 2009]

(a) -30 and -5 (b) -37 and -1
(c) -7 and 5 (d) 17.5 and -2

Sol. (c) $\because \quad \Sigma\lambda_i = \text{Trace}(A) = -2$

$$\Rightarrow \quad \lambda_1 + \lambda_2 = -2 \quad \ldots(i)$$

$$\Pi\lambda_i = |A| = -35$$

$$\Rightarrow \quad \lambda_1\lambda_2 = -35 \quad \ldots(ii)$$

Solving Eqs. (i) and (ii), we get

$$\lambda_1 \text{ and } \lambda_2 = 5, -7$$

Example 17. The eigen values and the corresponding eigen vectors of a 2×2 matrix are given by

Eigen value	Eigen vector
$\lambda_1 = 8$	$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$\lambda_2 = 4$	$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

The matrix is **[EC GATE 2006]**

(a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 6 \\ 6 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & 8 \\ 8 & 4 \end{bmatrix}$

Sol. (a) By property of eigen values, sum of diagonal elements should be equal to sum of values of λ.

So, $\Sigma \lambda_i = \lambda_1 + \lambda_2 = 8 + 4 = 12 = \text{Trace}(A)$

Only in choice (a), Trace $(A) = 12$.

or (a) $\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$

Example 18. The linear operation $L(x)$ is defined by the cross product $L(x) = bX_x$, where $b = [0\,1\,0]^T$ and $x = [x_1\ x_2\ x_3]^T$ are three dimensional vectors. The 3×3 matrix M of this operation satisfies $L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then the eigenvalues of M are **[EE GATE 2007]**

(a) $0, +1, -1$ (b) $1, -1, 1$

(c) $i, -i, 1$ (d) $i, -i, 0$

Sol. (d) $\because$ The cross product of $b = [0\ 1\ 0]^T$ and $x = [x_1\ \ x_2\ \ x_3]^T$

We can write above in the form

$$b \times X = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = x_3\,\hat{\mathbf{i}} + 0\,\hat{\mathbf{j}} - x_1\,\hat{\mathbf{k}}$$

$$= [x_3\ \ 0\ \ -x_1]$$

Now, we have $L(x) = b \times X = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

where, M is a 3×3 matrix

Let $$M = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

Now, $$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times X$$

$$\Rightarrow \quad \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

By matching LHS and RHS, we get

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}$$

So, $$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Now, we have to find the eigen values of M.

$$|M - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -\lambda(\lambda^2 - 0) + 1(0 - \lambda) = 0$$

$$\Rightarrow \quad \lambda^3 + \lambda = 0$$

$$\Rightarrow \quad \lambda = 0 \text{ and } \lambda^2 = -1$$

$$\Rightarrow \quad \lambda^2 = \pm i$$

Hence, $\lambda = 0, \pm i$

So, the eigen values are $0, i, -i$.

Example 19. The characteristic roots of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ are

(a) $1, -3, 7$ (b) $1, -4, 7$

(c) $0,\ 7,\ 3$ (d) None of these

Sol. (b) We have, $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$

So, the characteristic equation is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & -4-\lambda & 2 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

$$\Rightarrow \quad \lambda = 1, -4, 7$$

Hence, characteristic roots are $1, -4, 7$.

Example 20. The eigen vector of $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ are

(a) $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ (d) None of these

Sol. (a) $\because$ We have, $A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

Now, the characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$\Rightarrow \quad (\lambda - 1)(\lambda - 2)^2 = 0$$

$$\Rightarrow \quad \lambda = 1, 2, 2$$

Hence, the eigen values are 1, 2, 2.

Now, for eigen vectors

$$[A - \lambda I]\, X = 0 \quad \text{...(A)}$$

Putting $\lambda = 1$ in Eq. (i)

$$\begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

$$\Rightarrow \quad x_3 = -x_2 \quad \text{...(i)}$$

$$-x_1 + 2x_2 + x_3 = 0 \quad \text{...(ii)}$$

Let $x_3 = k \;\Rightarrow\; x_2 = -k$

So, from Eq. (ii)

$$-x_1 - 2k + k = 0 \;\Rightarrow\; x_1 = -k$$

Hence, $$X_1 = \begin{bmatrix} -k \\ -k \\ k \end{bmatrix} = -k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

or $$X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Now, putting $\lambda = 2$ in Eq. (i), we get

$$\begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & 2 & 2 \\ -1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to R_2 - R_1$

$R_1 \to -R_1$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Operating $R_2 \to -\frac{1}{2} R_2$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, operating $R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 - 2x_2 - 2x_3 = 0 \quad \text{...(iii)}$$

$$x_3 = 0 \quad \text{...(iv)}$$

Let $x_2 = k$

Then from Eqs. (iii) and (iv),

$$x_1 - 2k = 0$$

$$\Rightarrow \quad x_1 = 2k$$

Hence, $$X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k \\ k \\ 0 \end{bmatrix} = k \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

or $$X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$\because$ λ has two same values 2, 2.

Hence, X_3 is also equal to X_2

Hence, $$X_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},\; X_2 = X_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Example 21. The eigen vectors of the matrix $\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$ are written in the form $\begin{bmatrix} 1 \\ a \end{bmatrix}$ and $\begin{bmatrix} 1 \\ b \end{bmatrix}$ then what is $a + b$?

[ME GATE 2008]

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

Sol. (b) $\because$ $A = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

The characteristic equation is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) = 0$$

$$\Rightarrow \quad \lambda = 1, 2$$

Now, putting $\lambda = 1$, in

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = 0$$

$$2a = 0$$

$$\Rightarrow \quad a = 0$$

Again, putting $\lambda = 2$ in

$$\begin{bmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \qquad -1 + 2b = 0$$

$$2b = 1$$

$$\Rightarrow \qquad b = \frac{1}{2}$$

Hence, $a + b = 0 + \frac{1}{2} = \frac{1}{2}$

Example 22. How many of the following matrices have an eigen value 1?

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$$

[CS GATE 2008]

(a) One (b) Two (c) Three (d) Four

Sol. (a) For $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, the eigen values are

$$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad (1-\lambda)(0-\lambda) - 0 = 0$$

$$\Rightarrow \qquad \lambda = 0, 1$$

Now, for $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then eigen values are

$$\begin{vmatrix} 0-\lambda & 1 \\ 0 & 0-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \qquad -\lambda X - \lambda = 0$$

$$\Rightarrow \qquad \lambda = 0, 0$$

For $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, the eigen values are $\begin{vmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(1-\lambda) + 1 = 0$$

$$(1-\lambda)^2 + 1 = 0$$

$$\Rightarrow \qquad \lambda^2 - 2\lambda + 2 = 0$$

$$\Rightarrow \qquad \lambda = 1 \pm i$$

Now, for $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$, the eigen values are

$$\begin{vmatrix} -1-\lambda & 0 \\ 1 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)(-1-\lambda) - 0 = 0$$

$$\Rightarrow \qquad (-1-\lambda)^2 = 0$$

$$\Rightarrow \qquad \lambda = -1, -1$$

Hence, only one matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ has an eigen value of 1.

Intro Exercise 4

1. The eigen values of the matrix

$$[P] = \begin{bmatrix} 4 & 5 \\ 2 & -5 \end{bmatrix} \text{ are}$$

[CE GATE 2008]

(a) – 7 and 8 (b) – 6 and 5
(c) 3 and 4 (d) 1 and 2

2. An eigen vector of $P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ is

[EE GATE 2010]

(a) $[-1 \;\; 1 \;\; 1]'$ (b) $[1 \;\; 2 \;\; 1]'$
(c) $[1 \;\; -1 \;\; 2]'$ (d) $[2 \;\; 1 \;\; -1]'$

3. Given the matrix $\begin{bmatrix} -4 & 2 \\ 4 & 3 \end{bmatrix}$, the eigen vector is **[EC GATE 2005]**

(a) $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

4. For the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$, the eigen value corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$ is

[EC GATE 2006]

(a) 2 (b) 4
(c) 6 (d) 8

5. What are the eigen values of the following 2×2 matrix?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

[CS GATE 2005]

(a) – 1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and – 1

6. The eigen values of the matrix $\begin{bmatrix} 8 & 4 \\ 2 & 10 \end{bmatrix}$ are

(a) 6 and 12 (b) 10 and 7
(c) – 1 and 6 (d) 4 and – 6

7. The minimum and the maximum eigen values of the matrix $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are – 2 and 6 respectively, then what is the other eigen value? **[CE GATE 2007]**

(a) 5 (b) 3
(c) 1 (d) – 1

8. The two eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ are 3 and 2 respectively, then the value of other eigen value is

(a) 5 (b) 1 (c) 2 (d) – 1

9. The sum of the eigen values of the matrix given below $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is **[ME GATE 2004]**

(a) 5 (b) 7 (c) 9 (d) 18

10. The sum of the eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is

(a) 18 (b) 15 (c) 12 (d) 4

11. The sum of the eigen values of $\begin{bmatrix} -2 & 1 & 1 \\ -11 & 4 & 5 \\ -1 & 1 & 0 \end{bmatrix}$ is

(a) 4 (b) 2 (c) 6 (d) 0

12. The sum of eigen values of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is

(a) 9 (b) 5 (c) 10 (d) 12

13. The eigen values of A^5 when

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 6 & 1 \end{bmatrix} \text{ are}$$

(a) 243, 64, 1 (b) 1, 64, 81
(c) 1, 1024, 243 (d) 1, 4, 3

14. The sum and product of the eigen values of $\begin{bmatrix} 2 & 3 & -2 \\ -2 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ are respectively

(a) 5, 20 (b) 21, 5 (c) 5, 21 (d) 25, 30

15. If a square matrix A is real and symmetric, then the eigen values **[ME GATE 2007]**

(a) are always real
(b) are always real and positive
(c) are always real and non-negative
(d) occur in complex conjugate pairs

16. The eigen values of the following matrix $\begin{bmatrix} -1 & 3 & 5 \\ -3 & -1 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ are **[EC GATE 2009]**

(a) $3, 3+5i, 6-i$ (b) $-6+5i, 3+i, 3-i$
(c) $3+i, 3-i, 5+i$ (d) $3, -1+3i, -1-3i$

17. Consider the following matrix $A = \begin{bmatrix} 2 & 3 \\ x & y \end{bmatrix}$. If the eigen values are 4 and 8, then **[CS GATE 2010]**

(a) $x = 4, y = 10$ (b) $x = 5, y = 8$
(c) $x = -3, y = 9$ (d) $x = -4, y = 10$

18. The eigen values of a skew-symmetric matrix are **[EC GATE 2010]**

(a) always zero
(b) always pure imaginary
(c) either zero or pure imaginary
(d) always real

19. If A is 3×3 matrix and Trace $A = 9$, $|A| = 24$ and one eigen value is 2, then other eigen values are

(a) 2, 5 (b) 3, 4 (c) 1, 4 (d) 2, 6

20. The correct set of eigen values of $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$ is

(a) $-1, 2, 3$ (b) $1, 3, -1$
(c) $-1, 2, +2$ (d) $1, 2, 2$

21. The eigen values of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ are

(a) $8, 3, -7$ (b) $3, 0, 15$
(c) $3, 0, -7$ (d) $5, 7, 8$

22. If $A = \begin{bmatrix} 1 & 2 \\ -8 & 11 \end{bmatrix}$ has the eigen values 3 and 9, then the eigen values of A^3 are

(a) 3, 80 (b) 9, 27 (c) 27, 729 (d) 9, 81

23. One of the eigen values of matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is 5, then the corresponding eigen vector is

(a) $[1 \;\; 0 \;\; 2]'$ (b) $[3 \;\; 0 \;\; 0]'$
(c) $[4 \;\; 3 \;\; 2]'$ (d) $[3 \;\; -1 \;\; 2]'$

24. For some scalar λ if matrix A is such that $(A - \lambda I)$ is singular, then

(a) λ is a characteristic of A
(b) λ is not a characteristic root of A
(c) λ is zero
(d) None of the above

25. The characteristic roots of real Skew-Symmetric matrix are
(a) zero or pure imaginary
(b) zero or real
(c) zero
(d) None of the above

26. The characteristics roots of Skew-Symmetric matrix are
(a) zero or pure real (b) zero or pure imaginary
(c) complex numbers (d) None of these

27. If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the characteristic roots of matrix A, then the A^{-1} have the roots
(a) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ (b) $\lambda_1, \lambda_2, \ldots, \lambda_n$
(c) Zero (d) None of these

28. If the characteristic roots of the matrix A are zero, then A is
(a) singular matrix (b) non-singular matrix
(c) symmetric matrix (d) None of these

29. If $q = X^T AX, q$ is positive definite, then
(a) $|A| < 0$ (b) $|A| > 0$
(c) $|A| = 0$ (d) None of these

30. If $q = X^T AX$, and $A = [a_{ij}]$, then q is positive definite, if
(a) $a_{ii} > 0, i = 1, \ldots, n$ (b) $a_{ii} > 0, i = 1, \ldots, n$
(c) $a_{ii} < 0, i = 1, \ldots, n$ (d) None of these

31. If A and B are n-order square matrix, matrix A is non-singular, and $A^{-1}B$ and BA^{-1} have characteristic roots λ and β, then
(a) $\lambda = \beta$ (b) $\lambda \neq \beta$
(c) $\lambda > \beta$ (d) $\lambda < \beta$

32. If $\lambda_1, \ldots, \lambda_n$ are the characteristic root of matrix A, then A^4 have the characteristic roots
(a) $\lambda_1, \lambda_2, \ldots, \lambda_n$ (b) $\lambda_1^4, \lambda_2^4, \ldots, \lambda_n^4$
(c) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ (d) $\frac{1}{\lambda_1^4}, \frac{1}{\lambda_2^4}, \ldots, \frac{1}{\lambda_n^4}$

33. If A is a singular matrix, then its characteristic root are
(a) zero (b) unity
(c) non-zero (d) None of these

34. If A and B are n-order square matrix and AB and BA have characteristic roots λ and β, then
(a) $\lambda \neq \beta$ (b) $\lambda > \beta$
(c) $\lambda < \beta$ (d) $\lambda = \beta$

35. If A is an identity matrix then, its characteristic root is
(a) $\lambda = 1$ (b) $\lambda = 0$
(c) $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 1$ (d) $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$

36. If $\lambda_1 \ldots \lambda_n$ are eigen values of matrix A, then trace of A is
(a) $\lambda_1\lambda_2\lambda_3 \ldots \lambda_n$ (b) $\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n$
(c) $\frac{1}{\lambda_1\lambda_2\lambda_3 \ldots \lambda_n}$ (d) $\frac{1}{\lambda_1 + \lambda_2 + \lambda_3 + \ldots + \lambda_n}$

Answers with Solutions

1. (b) For eigen values $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 5 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = -6, 5$$

2. (b) For eigen values $|P - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 2, 3$$

So, for eigen vectors

At $\lambda = 1[A - 1I]X_1 = 0$

$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = 0$$

$$x_2 + 2x_3 = 0 \Rightarrow x_3 = 0, x_1 = k$$

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, at $\lambda = 2$

$$[A - 2I]X_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_3 = 0$$

$$-x_1 + x_2 = 0 \Rightarrow x_2 = x_1$$

$$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

At $\lambda = 3$

$$[A - 3I]X_3 = 0$$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad -2x_1 + x_2 = 0 \Rightarrow x_2 = 2x_1$

and $\quad -x_2 + 2x_3 = 0 \Rightarrow x_2 = 2x_3$

$\Rightarrow \quad 2x_1 = x_2 = 2x_3$

$$\Rightarrow \quad X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = [1 \quad 2 \quad 1]'$$

3. (b)

4. (c) For eigen values $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad (4-\lambda)^2 - 4 = 0$

$\Rightarrow \quad \lambda = -2, 6$

At $\quad \lambda = -2$

$[A + 2I]X_1 = 0$

$$\Rightarrow \quad \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 6x_1 + 2x_2 = 0$

$2x_1 + 6x_2 = 0$

$\Rightarrow \quad X_1 = \begin{bmatrix} k \\ -k \end{bmatrix}$ so this is not possible.

At $\lambda = 6$

$[A - 6I]X_2 = 0$

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 = x_2$

$$\Rightarrow \quad X_2 = \begin{bmatrix} k \\ k \end{bmatrix}$$

If $k = 101$

$$\Rightarrow \quad X_2 = \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

5. (b) **6.** (a) **7.** (b) **8.** (a)

9. (b) $\because \; A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\because$ The sum of main diagonal elements is the sum of eigen values so the sum of eigen values

$= a_{11} + a_{22} + a_{33}$

$= 1 + 5 + 1 = 7$

10. (a) **11.** (b) **12.** (c) **13.** (c)

14. (c) **15.** (c) **16.** (d)

17. (d) $\because \; 2 + y =$ sum of eigen values

$\Rightarrow \quad 2 + y = 4 + 8$

$y = 10$

and $\quad x =$ difference of eigen values

$= 4 - 8 = -4$

so $\quad x = -4, \; y = 10$

18. (c) $\because$ In skew-symmetric matrix the all diagonal elements are zero or purely imaginary.

Hence, the eigen values of a skew-symmetric matrix are either zero or purely imaginary.

19. (b) $\because \; \text{tr}(A) = 9$ and $|A| = 24$

$\lambda_1 = 2$

$\because \quad \lambda_1 + \lambda_2 + \lambda_3 = 9$

$\Rightarrow \quad \lambda_1 + \lambda_3 = 7 \quad$ ...(i)

and $\quad \lambda_1 \times \lambda_2 \times \lambda_3 = |A|$

$\Rightarrow \quad \lambda_2 \lambda_3 = 12 \quad$ ...(ii)

From Eqs. (i) and (ii),

$\lambda_2 = 3$ and $\lambda_3 = 4$

20. (d) **21.** (b)

22. (c) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of a matrix A then $\lambda_1^n, \lambda_2^n, \lambda_3^n, \ldots, \lambda_n^n$ are the eigen values of A^n.

Hence, the eigen values of A^3 are

3^3 and $9^3 = 27, 729$

23. (b) **24.** (a) **25.** (a) **26.** (b) **27.** (a)

28. (a) **29.** (b) **30.** (a) **31.** (a) **32.** (b)

33. (a) $\because$ The characteristic equation is $|A - \lambda I| X = 0$

$\Rightarrow \quad |AX| = |\lambda X|$

$\therefore \quad \lambda = 0$

$\Rightarrow \quad |AX| = 0$

$\Rightarrow \quad |A| = 0$

$\because \quad |X| \neq 0$

34. (d)

35. (a) $\because$ A is identity matrix, then it contains only diagonal elements, *i.e.*, unit.

$\because$ We know that the eigen values of diagonal matrix are always the product of diagonal element

i.e., $\quad (1-\lambda)(1-\lambda)\ldots(1-\lambda) = 0$

$\Rightarrow \quad 1 - \lambda = 0$

$\Rightarrow \quad \lambda = 1$

36. (b) $\because \; \lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigen values of given matrix and we know that the trace of any matrix is always the sum of eigen values or sum of diagonal elements

So, $\quad \text{trace}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

or

For a square matrix A of order n, if

$$|A-\lambda I| = (-1)^n (\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n)$$

be the characteristic polynomial, then the matrix equation

$$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots + a_n I = 0$$

is satisfied by $X = A$

i.e., $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$

Proof Let $A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$ be a square matrix of order n, then

$$A-\lambda I = \begin{bmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{bmatrix}$$

and let

$$|A-\lambda I| = (-1)^n (\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a^n) \quad \ldots\text{(i)}$$

The elements of $A-\lambda I$ are atmost of first degree in λ, so the elements of adj $(A-\lambda I)$ are the polynomials in λ of degree $n-1$ or less (since the cofactors of the elements of $A-\lambda I$ are at the most of degree $n-1$ in λ). Therefore, we can write adj $(A-\lambda I)$ as a matrix polynomial in λ as

$$\text{adj}\,(A-\lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1} \quad \ldots\text{(ii)}$$

where $B_0, B_1, B_2, \ldots, B_{n-1}$ are matrices of order $n \times n$.

Also, $A\,(\text{adj}\,A) = |A|.I = (\text{adj}\,A)\cdot A$

where I is an $n \times n$ identity matrix.

$$\Rightarrow \quad (A-\lambda I)\,\text{adj}\,(A-\lambda I) = |A-\lambda I|\,I \quad \ldots\text{(iii)}$$

$\therefore$ Substituting Eqs. (i) and (ii) in Eq. (iii), we obtain

$$(A-\lambda I)\cdot(B_0\lambda^{n-1} + B_1\lambda^{n-2} + B_2\lambda^{n-3} + \ldots + B_{n-2}\lambda + B_{n-1})$$
$$= (-1)^n (\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_n)I$$

Then equating the coefficients of like powers of λ on both sides, we get

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$AB_{n-1} = (-1)^n a_n I$$

Pre-multiplying the above equations by $A^n, A^{n-1}, A^{n-2}, \ldots, I$ respectively and adding, we get

$$0 = (-1)^n (A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I)$$
$$\Rightarrow \quad A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

where 0 is the null matrix of order $n \times n$. Hence, the theorem is verified.

Corollary : Inverse of a Matrix by Cayley-Hamilton Theorem

Let $|A-\lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \ldots + a_{n-1}\lambda + a_n$ is the characteristic polynomial of A.

So, we have

$$a_0 A^n + a_1 A^{n-1} + \ldots + a_{n-1}A + a_n I = 0 \quad \ldots\text{(i)}$$

Now, pre-multiplying Eq. (i) by A^{-1}, we get

$$a_0 A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-1} I + a_n A^{-1} = 0$$

$$\Rightarrow -a_n A^{-1} = a_0 A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-1} I$$

$$\Rightarrow A^{-1} = -\frac{1}{a_n}[a_0 A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \ldots + a_{n-1}]$$

Example 1. Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

Sol. We have, the characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

Operating $R_1 \to R_1 + R_2$

$$\begin{vmatrix} 1-\lambda & 1-\lambda & 0 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

Operating $\quad C_2 \to C_2 - C_1$

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -1 & 3-\lambda & -1 \\ 1 & -2 & 2-\lambda \end{vmatrix} = 0$$

On expanding through first row

$$(1-\lambda)[(3-\lambda)(2-\lambda)-2] = 0$$

$$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0 \quad \ldots\text{(i)}$$

This is characteristic equation.
Now, for Cayley-Hamilton theorem, matrix A must satisfy Eq. (i). So, on putting $\lambda = A$ we get,

$$A^3 - 6A^2 + 9A - 4I = 0 \quad \ldots\text{(ii)}$$

Now, $$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

On putting the values of A, A^2, A^3 and I in RHS of Eq. (ii), we get

$$A^3 - 6A^2 + 9A - 4I$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Hence, the Cayley-Hamilton theorem is verified.

Now, we have to find A^{-1}. So, on multiplying by A^{-1} to both sides of Eq. (i)

$$A^3A^{-1} - 6A^2A^{-1} + 9AA^{-1} - 4IA^{-1} = 0$$

$$\Rightarrow \quad A^2 - 6A + 9I - 4A^{-1} = 0$$

$$\Rightarrow \quad 4A^{-1} = A^2 - 6A + 9I$$

$$\Rightarrow \quad 4A^{-1} = +\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Hence, $$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Example 2. Find A^{-1} by Cayley-Hamilton theorem of $A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$.

(a) $\frac{1}{5}\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\frac{1}{10}\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$

(c) $\frac{1}{10}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$ (d) $\frac{1}{5}\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

Sol. (b) The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) - 12 = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda - 10 = 0$$

For the verification of Cayley-Hamilton theorem, put $\lambda = A$

$$\Rightarrow \quad A^2 - 3A - 10I = 0$$

Multiplying by A^{-1} on both sides

$$A^{-1}A^2 - 3A^{-1}A - 10A^{-1}I = 0 \quad [\because AA^{-1} = I;\ IA = A]$$

$$\Rightarrow \quad A - 3I - 10A^{-1} = 0$$

$$\Rightarrow \quad 10A^{-1} = A - 3I$$

$$= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 3-0 \\ 4-0 & 2-3 \end{bmatrix}$$

$$10A^{-1} = \begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 3 \\ 4 & -1 \end{bmatrix}$$

Example 3. The characteristic equation of a (3×3) matrix P is defined as

$$a(\lambda) = |\lambda I - P|$$
$$= \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$

If I denotes identity matrix, then the inverse of matrix P will be **[EE GATE 2008]**

(a) $P^2 + P + 2I$ (b) $P^2 + P + 1$
(c) $-(P^2 + P + 1)$ (d) $-(P^2 + P + 2I)$

Sol. (d) $\because$ Characteristic equation is

$$\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$

Then by Cayley-Hamilton theorem

$$P^3 + P^2 + 2P + I = 0$$

Multiplying by A^{-1} to both sides.

$$P^2 + P + 2I + P^{-1} = 0$$
$$\Rightarrow \quad P^{-1} = -P^2 - P - 2I$$
$$P^{-1} = -(P^2 + P + 2I)$$

Example 4. Let P be a 2×2 real orthogonal matrix and $\vec{\mathbf{X}}$ is a real vector $[X_1, X_2]$ with length $||\vec{\mathbf{x}}|| = (x_1^2 + x_2^2)^{1/2}$, then which one of the following is correct?

(a) $||P\vec{\mathbf{X}}|| \le ||\vec{\mathbf{x}}||$, where atleast one vector satisfies $||P\vec{\mathbf{X}}|| < ||\vec{\mathbf{x}}||$

(b) $||P\vec{\mathbf{X}}|| \le ||\vec{\mathbf{x}}||$, for all vectors $\vec{\mathbf{X}}$

(c) $||P\vec{\mathbf{X}}|| \ge ||\vec{\mathbf{x}}||$, where atleast one vector satisfies $||P\vec{\mathbf{X}}|| \ge ||\vec{\mathbf{x}}||$

(d) No relationship can be established between $||\vec{\mathbf{x}}||$ and $||P\vec{\mathbf{X}}||$ **[EE, GATE 2008]**

Sol. (b) Let an orthogonal matrix is

$$P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Now, by the property of orthogonal matrix

$$AA' = I$$

$$\Rightarrow \quad P\vec{\mathbf{X}} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}$$

$$\Rightarrow \quad |P\vec{\mathbf{X}}|$$
$$= \sqrt{(x_1\cos\theta - x_2\sin\theta)^2 + (x_1\sin\theta + x_2\cos\theta)^2}$$
$$= \sqrt{x_1^2 + x_2^2}$$
$$= (x_1^2 + x_2^2)^{1/2}$$
$$= ||\vec{\mathbf{x}}|| \text{ for all vector } \vec{\mathbf{x}}.$$

Similar Matrices

Matrices A and B are said to be similar if there exists non-singular matrix P such that

$$A = P^{-1}BP$$

or A is said to be similar to B if there exists non-singular matrix P such that

$$A = P^{-1}BP, \text{ denoted by } A \cong B.$$

Properties of Similar Matrix

(a) If $A \cong B$ *i.e.,* we have a non-singular matrix p such that

$$A = P^{-1}BP, \quad |P| \neq 0$$

Then, A and B may be singular.

(b) If A and B are similar matrices, then

$$\text{ch}_A(x) = \text{ch}_B(x)$$

not conversely.

Hence, A and B have same eigen value, same trace and same determinant.

(c) If A and B are similar matrices, then

$$m_A(x) = m_B(x)$$

Hence, $\rho(A) = \rho(B)$, not conversely.

For example, the matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are not similar as $\rho(A) = 2$ and $\rho(B) = 1$, while

$$\text{ch}_A(x) = x^4 = \text{ch}_B(x)$$
$$m_A(x) = x^2 = m_B(x)$$
$$\text{Trace}(A) = \text{Trace}(B)$$
$$\det(A) = \det(B)$$

Eigen value of A = Eigen value of B.

Diagonalization of Matrices

Let A be a square matrix, then the matrix A is said to be diagonalizable if it is similar to a diagonal matrix, *i.e.,* there exist an invertible matrix P such that

$P^{-1}AP = D$, where D is a diagonal matrix.

> Since, similar matrices have the same eigen values, the diagonal elements of D are the eigen values of A.

Theorem If a square matrix A (order $n \times n$) has n linearly independent eigen vectors, then a matrix P can be found such that $P^{-1}AP$ is always a diagonal matrix, where $P = [X_1, X_2, \ldots, X_n]$ and X_i are eigen vectors.

Proof We prove this theorem for a square matrix of order 3×3 and the proof can be easily extended to matrices of higher orders.

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be a 3×3 matrix. ...(i)

Let λ_1, λ_2 and λ_3 are its eigen values and X_1, X_2 and X_3 the corresponding eigen vectors of matrix A defined as follows

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \; X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

For the eigen value λ_1, the eigen vector is

$$(a_1 - \lambda_1)x_1 + b_1 y_1 + c_1 z_1 = 0$$
$$a_2 x_1 + (b_2 - \lambda_1)\, y_1 + c_2 z_1 = 0$$
$$a_3 x_1 + b_3 y_1 + (c_3 - \lambda_1)\, z_1 = 0$$

$$\Rightarrow \quad \left.\begin{aligned} a_1 x_1 + b_1 y_1 + c_1 z_1 &= \lambda_1 x_1 \\ a_2 x_1 + b_2 y_1 + c_2 z_1 &= \lambda_1 y_1 \\ a_3 x_1 + b_3 y_1 + c_3 z_1 &= \lambda_1 z_1 \end{aligned}\right\} \quad \ldots\text{(ii)}$$

and similarly for λ_2 and λ_3, we obtain

$$\Rightarrow \quad \left.\begin{aligned} a_1 x_2 + b_1 y_2 + c_1 z_2 &= \lambda_2 x_2 \\ a_2 x_2 + b_2 y_2 + c_2 z_2 &= \lambda_2 y_2 \\ a_3 x_2 + b_3 y_2 + c_3 z_2 &= \lambda_2 z_2 \end{aligned}\right\} \quad \ldots\text{(iii)}$$

$$\text{and} \quad \left.\begin{aligned} a_1 x_3 + b_1 y_3 + c_1 z_3 &= \lambda_3 x_3 \\ a_2 x_3 + b_2 y_3 + c_2 z_3 &= \lambda_3 y_3 \\ a_3 x_3 + b_3 y_3 + c_3 z_3 &= \lambda_3 z_3 \end{aligned}\right\} \quad \ldots\text{(iv)}$$

Now, let us consider the matrix P, such that its columns are the eigen vectors of A.

i.e., $$P = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

Then, $$AP = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 x_1 + b_1 y_1 + c_1 z_1 & a_1 x_2 + b_1 y_2 + c_1 z_2 & a_1 x_3 + b_1 y_3 + c_1 z_3 \\ a_2 x_1 + b_2 y_1 + c_2 z_1 & a_2 x_2 + b_2 y_2 + c_2 z_2 & a_2 x_3 + b_2 y_3 + c_2 z_3 \\ a_3 x_1 + b_3 y_1 + c_3 z_1 & a_3 x_2 + b_3 y_2 + c_3 z_2 & a_3 x_3 + b_3 y_3 + c_3 z_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \lambda_3 x_3 \\ \lambda_1 y_1 & \lambda_2 y_2 & \lambda_3 y_3 \\ \lambda_1 z_1 & \lambda_2 z_2 & \lambda_3 z_3 \end{bmatrix} \text{ [Using Eqs. (ii), (iii) and (iv)]}$$

Making use of Eqs. (ii), (iii) and (iv); we have

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = PD$$

where, $$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

is a diagonal matrix of order 3×3.

$$\Rightarrow \quad AP = PD \quad \ldots\text{(v)}$$

Pre-multiplying by B^{-1}, we get

$$P^{-1}AP = P^{-1}PD = ID = D$$

which implies that A is similar to D. Therefore, the matrix of eigen vectors P reduces matrix A to its diagonal form.

Post multiplying Eq. (v) by P^{-1}, we get

$$A = PDP^{-1}$$

This result can be generalized to $n \times n$ order matrix. Here, the matrix P is called the modal matrix of A and D is called the spectral matrix of A.

Remark The square matrix P, which diagonalises A, is obtained by grouping the eigen vectors of A into square matrix and then resulting diagonal matrix has the eigen values of A as its diagonal elements.

Properties of Diagonal Matrices

(a) A matrices is diagonalizable iff sum of geometric multiplicity is equal to the order of A.

(b) A matrix is diagonalizable iff for every eigen value geometric multiplicity is equal to algebraic multiplicity.

(c) If all the eigen values of a matrix are distinct, then matrix can be diagonalizable not conversely.

(d) A matrix A can be diagonalizable iff its minimal polynomial $m_A(x)$ does not contain repeated linear factor *i.e.,* minimal polynomial is product of distinct linear terms *i.e.,*

$$m_A(x) = \Pi(x - \lambda_i)$$
$$\lambda_i \in F$$

(e) If all the entries of A are equal, then A is diagonalizable. For $A = [a_{ij}]_{n\times n}$ and $a_{ij} = c\ \forall\ i, j$ ch $(x) = x^{n-1}(x - nc) \Rightarrow m_A(x) = x(x - nc)$

(f) Every idempotent matrix can be diagonalizable.

(g) Every involutory matrix can be diagonalizable.

(h) Every real symmetric matrix is diagonalizable.

(i) Every Hermitian matrix can be diagonalizable.

(j) Nilpotent matrix can never be diagonalizable unless it is zero matrix.

Power of a Matrix by the Method of Diagonalization

Let A be a square matrix of order $n \times n$. Then we can find a non-singular matrix P such that

$$D = P^{-1}AP$$

$$\Rightarrow \quad D^2 = (P^{-1}AP)(P^{-1}AP)$$
$$= (P^{-1}APP^{-1}AP)$$
$$D^2 = P^{-1}A^2P \qquad (\because PP^{-1} = I)$$

Similarly, $\quad D^3 = P^{-1}A^3P$

....

and $\quad D^n = P^{-1}A^nP \qquad \ldots(i)$

Pre-multiply Eq. (i) by P and post-multiply by P^{-1}, we have

$$PD^nP^{-1} = PP^{-1}A^nPP^{-1} = A^n, \text{ which gives } A^n.$$

Example 5. Investigate whether A is similar to P, where

$$A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} \text{ and } P = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Sol. We know if two matrices are similar then there exists an invertible matrix B such that

$$A = B^{-1}PB \quad \text{or} \quad BA = PB$$

Let $\quad B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

Now, we shall determine the values of p, q, r and s such that $BA = PB$, and check if B is non-singular.

$$\therefore \quad \begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 5p - 2q & 5p \\ 5r - 2s & 5r \end{bmatrix} = \begin{bmatrix} p + 2r & q + 2s \\ -3p + 4r & -3q + 4s \end{bmatrix}$$

Here, we obtain following system of equations when we equate the corresponding elements.

$5p - 2q = p + 2r$ or $4p - 2q - 2r = 0$ ...(i)

$5p = q + 2s$ or $5p - q - 2s = 0$ ...(ii)

$5r - 2s = -3p + 4r$ or $r - 2s + 3p = 0$ ...(iii)

$5r = -3q + 4s$ or $5r + 3q - 4s = 0$ ...(iv)

Solving Eqs. (i), (ii), (iii) and (iv), we get

$$p = 1, q = 1, r = 1 \text{ and } s = 2$$

$$\Rightarrow B = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \text{ which is a non-singular matrix.}$$

Hence, the matrix A and P are similar.

Example 6. Reduce the matrix A into a diagonal matrix where, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Sol. Proceding as above its eigen values are calculated to be 0, 3, 15. Also the corresponding vectors are

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ and } X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

whence matrix $B = [X_1 \quad X_2 \quad X_3]$,

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Also, $\quad B^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$\Rightarrow \quad B^{-1}AB = \frac{1}{9}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 45 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

Example 7. Find a matrix B which reduces to the diagonal form by the transformation BAB^{-1}, where

$$A=\begin{bmatrix}-1 & 1 & 2\\ 0 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}$$

Sol. The characteristic equation is given by $\det(A-\lambda I)=0$.

$$\Rightarrow \quad \begin{vmatrix}-1-\lambda & 1 & 2\\ 0 & -2-\lambda & 1\\ 0 & 0 & -3-\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad (-1-\lambda)(-2-\lambda)(-3-\lambda)=0$$

$$\Rightarrow \quad \lambda=-1,\ \lambda=-2, \lambda=-3$$

For $\lambda=-1$, the corresponding eigen vector is given by

$$\begin{bmatrix}-1+1 & 1 & 2\\ 0 & -2+1 & 1\\ 0 & 0 & -3+1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned}x_2+2x_3&=0\\ -x_2+x_3&=0\\ -2x_3&=0\end{aligned}\right] \Rightarrow \quad x_2=x_3=0$$

whence the eigen vector may be taken as $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

For $\lambda=-2$, then eigen vector is given by

$$\begin{bmatrix}-1+2 & 1 & 2\\ 0 & -2+2 & 1\\ 0 & 0 & -3+2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1 & 1 & 2\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{By } R_3 \to R_3+R_2$$

$$\Rightarrow \quad x_1+x_2+2x_3=0,\ x_3=0$$

These equations are satisfied by taking $x_1=1, x_2=-1, x_3=0$.

Hence, the eigen vector is $\begin{pmatrix}1\\-1\\0\end{pmatrix}$.

For $\lambda=-3$, the corresponding eigen vector is given by

$$\begin{bmatrix}-1+3 & 1 & 2\\ 0 & -2+3 & 1\\ 0 & 0 & -3+3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}2 & 1 & 2\\ 0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad 2x_1+x_2+2x_3=0,\ x_2+x_3=0$$

Let us assume $x_1=1$, so that $x_2=2, x_3=-2$

Then the third eigen vector is $\begin{pmatrix}1\\2\\-2\end{pmatrix}$.

$$\Rightarrow \quad B=\begin{bmatrix}1 & 1 & 1\\ 0 & -1 & 2\\ 0 & 0 & -2\end{bmatrix}$$

$\Rightarrow$ This reduces BAB^{-1} to diagonal form.

Intro Exercise 5

1. The square matrix A is defined as

$A=\begin{bmatrix}1 & 2 & -2\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$, the diagonal matrix D of A is

(a) $\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$ (b) $\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 2\end{bmatrix}$

(c) $\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 3\end{bmatrix}$ (d) $\begin{bmatrix}3 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 8\end{bmatrix}$

2. The matrix $A=\begin{bmatrix}3 & -1 & 1\\ -1 & 5 & -1\\ 1 & -1 & 3\end{bmatrix}$ then the value of A^4 is

(a) $\begin{bmatrix}250 & -400 & 235\\ -400 & 87 & -400\\ 235 & -400 & 279\end{bmatrix}$

(b) $\begin{bmatrix}251 & -405 & 235\\ -405 & 81 & -405\\ 235 & -405 & 251\end{bmatrix}$

(c) $\begin{bmatrix} 270 & -425 & 255 \\ -425 & 100 & -425 \\ 255 & -425 & 283 \end{bmatrix}$

(d) None of the above

3. The diagonal matrix of $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ is

(a) $\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 12 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

4. The value of A^5 for the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 2344 & 781 \\ 2343 & 782 \end{bmatrix}$ (b) $\begin{bmatrix} 781 & 2344 \\ 782 & 2343 \end{bmatrix}$

(c) $\begin{bmatrix} 781 & 2344 \\ 2343 & 782 \end{bmatrix}$ (d) None of these

5. The diagonal matrix of $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & -\sqrt{5} \end{bmatrix}$ (b) $\begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} \sqrt{7} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$

6. The model matrix P of $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 7 \\ -5 & -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & -8 & 5 \\ -8 & 7 & 0 \\ 5 & 0 & -6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 7 & 6 & -5 \\ 6 & 4 & -3 \\ -5 & -3 & 1 \end{bmatrix}$

Answers with Solutions

1. (c) The characteristic roots are $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ -1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 2-\lambda \\ -1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\{(2-\lambda)\, X - \lambda + 1\} - 2\{-\lambda + 1\} - 2\{-1 + 2 - \lambda\} = 0$$

$$\Rightarrow (1-\lambda)[-2\lambda + \lambda^2 + 1] - 2(1-\lambda) - 2(1-\lambda) = 0$$

$$\Rightarrow (1-\lambda)[\lambda^2 - 2\lambda + 1 - 4] = 0$$

$$\Rightarrow \lambda = 1, -1, 3$$

Now, eigen vectors

At $\lambda = 1 \Rightarrow [A - I]X_1 = 0$

$$\begin{bmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 - x_3 = 0 \Rightarrow x_2 = x_3$$

$$\Rightarrow x_1 + x_2 + x_3 = 0 \Rightarrow x_1 = -2x_3$$

$$\Rightarrow X_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

At $\lambda = -1 \Rightarrow [A + I]X_2 = 0$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ x_1 + 3x_2 + x_3 &= 0 \end{aligned}\right\}$$

$$\Rightarrow 2x_1 + 4x_2 = 0$$

$$\Rightarrow 2x_2 = -x_1 \Rightarrow x_3 = x_2$$

$$\Rightarrow X_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

At $\lambda = 3 \Rightarrow [A - 3I]X_3 = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \left.\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 - x_2 - 3x_3 &= 0 \end{aligned}\right\}$$

$$\Rightarrow \quad -2x_2 - 2x_3 = 0$$

$$\Rightarrow \quad x_3 = -x_2$$

$$\Rightarrow \quad x_1 = 2x_2$$

$$\Rightarrow \quad X_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

So, model matrix

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -2 & -2 & -2 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

So, diagonal matrix of

$$A = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

2. (b)

3. (b) $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

For eigen values $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = 0, 3, 15$$

So, diagonal matrix

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

4. (a) **5.** (a) **6.** (c)

Unit Exercise 1

(1 Mark Questions)

1. If $A = \begin{bmatrix} 2 & 0 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -3 \\ -6 & 2 \end{bmatrix}$ are such that $4A + 3X = 5B$, then X is equal to

(a) $\begin{bmatrix} 4 & -5 \\ -6 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 5 \\ -6 & -2 \end{bmatrix}$

(c) $\begin{bmatrix} -4 & 5 \\ 6 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 4 & -5 \\ -6 & -2 \end{bmatrix}$

2. If $A - 2B = \begin{bmatrix} 1 & -2 \\ 3 & 0 \end{bmatrix}$ and $2A - 3B = \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}$, then B is equal to

(a) $\begin{bmatrix} 6 & -4 \\ -3 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 6 \\ -3 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & -6 \\ 3 & -3 \end{bmatrix}$ (d) $\begin{bmatrix} -4 & -6 \\ -3 & -3 \end{bmatrix}$

3. If $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$, then x and y are

(a) $-2, 8$ (b) $2, -8$

(c) $3, -6$ (d) $-3, 6$

4. If $A_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then $(A_\alpha)^2$ is equal to

(a) $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin^2\alpha & \cos^2\alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$

(c) $\begin{bmatrix} 2\cos\alpha & 2\sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ (d) $\begin{bmatrix} \cos^2\alpha & \sin^2\alpha \\ -\sin^2\alpha & \cos^2\alpha \end{bmatrix}$

5. The eigen vector of the matrix

$$\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \alpha \neq 0 \text{ is}$$

[CS GATE 1993]

(a) $(0, 0, \alpha)$ (b) $(0, 0, 1)$

(c) $(\alpha, 0, 0)$ (d) $(0, \alpha, 0)$

6. Let A and B be real symmetric matrices of size $n \times n$. Then which one of the following is true? **[CS GATE 1994]**

(a) $AA' = 1$ (b) $A = A^{-1}$

(c) $AB = BA$ (d) $(AB)' = BA$

7. The rank of the matrix $\begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$ is **[CS GATE 1994]**

(a) zero (b) 1

(c) 2 (d) 3

8. In a compact single dimensional array representation for lower triangular matrices of size $n \times n$, non-zero elements of each row are stored one after another starting from the first row, the index of the (i, j)th element of the lower triangular matrix in this new representation is **[CS GATE 1994]**

(a) $i + j$ (b) $i + j - 1$

(c) $j + \frac{i(i-1)}{2}$ (d) $i + \frac{j(j-1)}{2}$

9. The rank of the following $(n+1) \times (n+1)$ matrix where a is real number

$$\begin{bmatrix} 1 & a & a^2 & \dots & a^n \\ 1 & a & a^2 & \dots & a^n \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & a & a^2 & \dots & a^n \end{bmatrix} \text{ is}$$

[CS GATE 1995]

(a) 1 (b) 2

(c) n (d) Depends on 'a'

10. If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ be such that $A + A' = I$, then α is equal to

(a) π (b) $\frac{\pi}{3}$

(c) π (d) $\frac{2\pi}{3}$

11. If $A = \begin{bmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{bmatrix}$ is singular, then k is equal to

(a) $\frac{16}{3}$ (b) $\frac{34}{5}$

(c) $\frac{33}{2}$ (d) None of these

12. If $A = \begin{bmatrix} 2x & 0 \\ x & x \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$, then x is equal to

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) -2

13. If A and B are square matrices of the same order, then $(A+B)(A-B)$ is equal to
(a) $(A^2 - B^2)$ (b) $A^2 + AB - BA - B^2$
(c) $A^2 - AB + BA - B^2$ (d) None of these

14. If A and B are square matrices of the same order, then $(A+B)^2$ is equal to
(a) $A^2 + 2AB + B^2$ (b) $A^2 + AB + BA + B^2$
(c) $A^2 + 2BA + B^2$ (d) None of these

15. Let $Ax = b$ be a system of linear equations where A is an $m \times n$ matrix and b is an $m \times 1$ column vector and X is an $n \times 1$ column vector of unknown. Which of the following is false? **[CS GATE 1995]**
(a) The system has a solution if and only if, both A and the augmented matrix $[Ab]$ have the same rank
(b) If $m < n$ and b is zero vector, then the system has infinitely many solutions
(c) If $m = n$ and b is non-zero vector, then the system has a unique solution
(d) The system will have only a trivial solution when $m = n$, b is the zero vector and rank $(A) = n$

16. Consider the following statements :
S_1 : The sum of two singular $n \times n$ matrices may be non-singular.
S_2 :The sum of two $n \times n$ non-singular matrices may be singular.
Which of the following statements is true? **[CS GATE 2001]**
(a) S_1 and S_2 both are true
(b) S_1 is true and S_2 is false
(c) S_1 is false, S_2 is true
(d) S_1 and S_2 both are false

17. The number of different $n \times n$ symmetric matrices with each element being either 0 or 1, is
(**Note** Power $(2, x)$ is same as 2^x) **[CS GATE 2004]**
(a) Power $(2, n)$ (b) Power $(2, n^2)$
(c) Power $\left(2, \dfrac{n^2+n}{2}\right)$ (d) Power $\left(2, \dfrac{n^2-n}{2}\right)$

18. Let A, B, C and D be $n \times n$ matrices each with non-zero determinant. If $ABCD = 1$, then B^{-1} is **[CS GATE 2004]**
(a) ADC (b) CDA
(c) $D^{-1}C^{-1}A^{-1}$ (d) Does not exist

19. A square matrix is singular whenever **[CS GATE 1987]**
(a) the rows are linearly independent
(b) the columns are linearly independent
(c) the rows are linearly dependent
(d) None of the above

20. If A and B are idempotent matrix, then AB is idempotent, if
(a) $AB = BA$ (b) $(AB)^T = B^T A^T$
(c) $AB \neq BA$ (d) None of these

21. The tranjugate of a matrix $\begin{bmatrix} 1+i & i \\ 2 & 1-i \end{bmatrix}$ is
(a) $\begin{bmatrix} 1-i & 2 \\ -i & 1+i \end{bmatrix}$ (b) $\begin{bmatrix} 1+i & 2 \\ i & 1-i \end{bmatrix}$
(c) $\begin{bmatrix} i & 2 \\ 1-i & 1+i \end{bmatrix}$ (d) $\begin{bmatrix} 1-i & 2 \\ 3 & 2-i \end{bmatrix}$

22. If A and B are idempotent matrices, then $A + B$ will be idempotent, iff
(a) $AB = BA =$ zero matrix
(b) $AB =$ zero matrix
(c) $BA =$ zero matrix
(d) None of the above

23. If A is any n-order square matrix and k is any scalar, then
(a) $|kA| = k^n |A|$ (b) $|kA| = k|A|$
(c) $|kA| = k^2|A|$ (d) None of these

24. The diag (1, 1, …, 1) is
(a) idempotent matrix
(b) rectangular matrix
(c) non-symmetric matrix
(d) None of the above

25. The solution of
$$3x + 7y + 8z = -13$$
$$2x + 9z = -5$$
$$-4x + y - 26z = 2, \text{ is}$$
(a) $x = -7, y = 0, z = 2$
(b) $x = 0, y = 0, z = 3$
(c) $x = -7, y = 0, z = 1$
(d) None of the above

26. If A is a square matrix of order n and det $(A) \neq 0$, then rank A is equal to
(a) zero (b) 1 (c) $\propto$ (d) n

27. If $A = \begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \\ 0 & 0 & 0 & k \end{bmatrix}$, then det (A) is
(a) $aehk$ (b) zero
(c) $a + e + h + k$ (d) None of these

28. If $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, then det (A) is

(a) 125 (b) 15
(c) zero (d) None of these

29. The characteristic equation and eigen value for matrix $A = \begin{bmatrix} 3-4i & -7 \\ 5i & 6+2i \end{bmatrix}$ are

(a) $\lambda^2 \neq 11\lambda + 18 = 0, 9, 2$
(b) $\lambda^2 - 2i\lambda + 8 = 0, 3, 4$
(c) $\lambda^2 + 6\lambda + 2 = 0, 1, 3$
(d) None of these

30. The value of the determinant $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix}$, is

(a) abc (b) $a^2b^2c^2$
(c) $bc + ca + ab$ (d) zero

31. The value of the determinant $\begin{vmatrix} m & n & p \\ p & m & n \\ n & p & m \end{vmatrix}$ is

(a) $m^3 + n^3 + p^3$ (b) $3mnp$
(c) $m^3 + n^3 + p^3 - 3mnp$ (d) zero

32. If A is a square matrix of order n, then A^{-1} exists if

(a) rank $A = 0$ (b) rank $A = n$
(c) rank $A < n$ (d) None of these

33. If $A = \text{diag}\,(a_{11} \ldots a_{nn})$, then A^{-1} is

(a) diag $(a_{11} \ldots a_{nn})$ (b) diag (1, 11, …, 1)
(c) diag $\left(\frac{1}{a_{11}}, \ldots, \frac{1}{a_{nn}}\right)$ (d) diag (0, 0, …, 0)

34. The rank of the matrix $\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}$ is

(a) 4 (b) 3 (c) 2 (d) 1

35. The solution of

$5x - 3x_2 = 37$
$-2x + 7x_2 = -38$ is

(a) $x_1 = 5, x_2 = -4$ (b) $x_1 = 2, x_2 = 2$
(c) $x_1 = 3, x_2 = 4$ (d) None of these

36. If Hermitian matrix A is real, then matrix A is

(a) Skew-Hermitian too (b) Skew-symmetric too
(c) symmetric too (d) None of these

37. The characteristic equation for the unitary matrix of matrix $A = \begin{bmatrix} 3+4i & -5i \\ -7 & 6-2i \end{bmatrix}$ is

(a) $\lambda^2 - i\lambda - 1 = 0$ (b) $\lambda^2 + i\lambda = 0$
(c) $\lambda^2 + i\lambda + 1 = 0$ (d) None of these

38. The matrix $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$ has eigen values

(a) 8, 8 (b) 8, 3
(c) 8, 2 (d) 0, 0

39. The Cramer's rule on system of equations $AX = B$, $X \neq 0$ does not apply when

(a) A is singular matrix
(b) B is singular matrix
(c) A is non-singular
(d) B is non-singular

40. If zero is the characteristic root of matrix A, then

(a) A is singular (b) A is non-singular
(c) A is identity (d) None of these

41. If A is a square matrix of order n, the inverse A^{-1}, if

(a) 0 is an eigen value of A
(b) 0 is not an eigen value of A
(c) 1 is an eigen value of A
(d) 1 is not an eigen value of A

42. The inverse of a square matrix A exists, if

(a) its columns are independent
(b) its columns are dependent
(c) $|A| = 0$
(d) None of the above

43. $A = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$, A is orthogonal, then

(a) $|A| = \pm 1$ (b) $|A| = 0$
(c) $|A^T| \neq |A|$ (d) None of these

44. The eigen values of Hermitian matrix are

(a) real (b) complex
(c) pure imaginary (d) None of these

45. A square matrix of order n, $\hat{\mathbf{A}}$ is similar to square matrix of order nA, then for same non-singular n-order square matrix

(a) $\hat{\mathbf{A}} \neq P^{-1}AP$ (b) $\hat{\mathbf{A}} = P^{-1}AP$
(c) $\hat{\mathbf{A}} = P^{-1}A$ (d) $\hat{\mathbf{A}} = PP^{-1}A$

46. If row vectors of a square matrix A are linearly dependent, then

(a) $|A|=0$ (b) $|A|\neq 0$

(c) $|A|=C$ (d) None of these

47. If $|A|\neq 0$, then

(a) $|\text{adj } A|=|A|^{n-1}$ (b) $|\text{adj } A|=|A|^{n}$

(c) $|\text{adj } A|=0$ (d) None of these

48. If A, B and C are three matrices, then

(a) $|ABC|=|A||B||C|$

(b) $|ABC|=|AB|C$

(c) $|ABC|=|A||BC|$

(d) None of the above

49. If row vectors of a non-zero square matrix A are linearly independent, then

(a) $|A|=0$ (b) $|A|\neq 0$

(c) $|A|=n$ (d) None of these

50. If A and B are two square matrices of same order

(a) $|AB|=|BA|$ (b) $|AB|\neq|B||A|$

(c) $|AB|\neq|BA|$ (d) None of these

51. The eigen values of matrix $A=\begin{bmatrix}1 & 2\\ 4 & 3\end{bmatrix}$ are 5 and -1.

Then, the eigen values of $-2A+3I$ (I is a 2×2 identity matrix) are **[CS GATE 2009]**

(a) -7 and 5 (b) 7 and -5

(c) $-\frac{7}{5}$ and $\frac{1}{5}$ (d) $\frac{1}{7}$ and $-\frac{1}{2}$

52. $x=[x_1 x_2 \ldots x_n]^T$ is an n-tuple non-zero vector. The $n\times n$ matrix $v=xx^T$ **[EE GATE 2007]**

(a) has rank zero (b) has rank 1

(c) is orthogonal (d) has rank n

53. Let A be an $n\times n$ real matrix such that $A^2=1$ and y be an n-dimensional vector. Then, the linear system of equations $Ax=y$ has **[IE GATE 2007]**

(a) no solution

(b) a unique solution

(c) more than one but finitely many independent solutions

(d) infinitely many independent solutions

54. A is a 3×4 real matrix and $Ax=b$ is an inconsistent system of equations. The highest possible rank of A is **[ME GATE 2005]**

(a) 1 (b) 2

(c) 3 (d) 4

55. The necesssary condition to diagonalise a matrix is that **[IE GATE 2001]**

(a) its eigen value should be distinct

(b) its eigen vectors should be independent

(c) its eigen value should be real

(d) the matrix is non-singular

56. The inverse of the matrix $\begin{bmatrix}0.2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.5\end{bmatrix}$ is **[CS GATE 2002]**

(a) $\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 5\end{bmatrix}$ (b) $\begin{bmatrix}-0.2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -0.5\end{bmatrix}$

(c) $\begin{bmatrix}5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 2\end{bmatrix}$ (d) $\begin{bmatrix}0.5 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0.2\end{bmatrix}$

57. The value of the following determinant

$$\begin{vmatrix}1 & 0 & 0 & 0 & 0\\ -2 & 2 & 0 & 0 & 0\\ 3 & 5 & 3 & 0 & 0\\ -1 & 4 & 7 & 4 & 0\\ -5 & -6 & 2 & 1 & 1\end{vmatrix}$$

is **[CS GATE 2001]**

(a) 24 (b) 32

(c) -112 (d) zero

Unit Exercise 2

(2 Marks Questions)

1. The matrix $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$ is

(a) idempotent (b) orthogonal
(c) nilpotent (d) None of these

2. If A is an invertible matrix and $A^{-1} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, then A is equal to

(a) $\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{5} & \frac{1}{6} \end{bmatrix}$

(c) $\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$ (d) None of these

3. If $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & i & i \\ 0 & 0 & 0 & -i \end{bmatrix}$ the matrix A^k, calculated by the use of Cayley-Hamilton theorem or otherwise is **[CS GATE 1996]**

(a) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) None of these

4. The determinant of the matrix

$$\begin{bmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{is}$$

[CS GATE 1997]

(a) 11 (b) – 48
(c) 0 (d) – 24

5. Let $a = (a_{ij})$ be an n-rowed square matrix and I_{12} be the matrix obtained by interchanging the first and second rows of the n-rowed identity matrix. Then AI_{12} is such that its first **[CS GATE 1997]**

(a) row is the same as its second row
(b) row is the same as the second row of A
(c) column is the same as the second column of A
(d) row is all zero

6. The rank of the matrix given below is

$$\begin{bmatrix} 1 & 4 & 8 & 7 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 3 & 12 & 24 & 2 \end{bmatrix}$$

[CS GATE 1998]

(a) 3 (b) 1
(c) 2 (d) 4

7. Consider the following determinant :

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Which of the following is a factor of Δ? **[CS GATE 1998]**

(a) $a+b$ (b) $a-b$
(c) $a+b+c$ (d) abc

8. Let P be a matrix of order $m \times n$ and Q be matrix of order $n \times P, n \neq P$. If rank $(P) = n$ and rank $(Q) = P$, then rank (PQ)

(a) n (b) P
(c) nP (d) $n+P$

9. Let P and Q be square matrices such that $PQ = I$, then zero is an eigen value of **[GATE 1999]**

(a) P but not of Q (b) Q but not of P
(c) both P and Q (d) neither P nor Q

10. The determinant of the matrix

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 8 & 1 & 7 & 2 \\ 2 & 0 & 2 & 0 \\ 9 & 0 & 0 & 1 \end{bmatrix} \text{is}$$

[GATE 2000]

(a) 4 (b) 0
(c) 15 (d) 20

11. The rank of the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is **[GATE 2001]**

(a) 4 (b) 1
(c) 2 (d) 0

12. Obtain the eigen values of the matrix

$$A = \begin{bmatrix} 1 & 2 & 34 & 49 \\ 0 & 2 & 43 & 94 \\ 0 & 0 & -2 & 104 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$ **[GATE 2002]**

(a) – 2, – 1, 1, 2 (b) – 5, – 3, 2,4
(c) 1, 3, 5, 7 (d) – 2, 0, 1, 4

13. Consider the following system of linear equations :

$$\begin{bmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \alpha \\ 5 \\ 7 \end{bmatrix}$$

Notice that the second and third columns of the coefficient matrix are linearly dependent. For how many values of α does this system of equations have infinite many solutions? **[GATE 2002]**

(a) 0 (b) 1
(c) 2 (d) Infinitely many

14. In an $M \times N$ matrix such that all non-zero entries are covered in a row and b column. Then the maximum number of non-zero entries, such that no two are on the same row or column, is **[CS GATE 2004]**

(a) $\leq a + b$ (b) $\leq \max(a, b)$
(c) $\leq \min\{M - a, N - b\}$ (d) $\leq \min\{a, b\}$

15. What are the eigen values of the following 2×2 matrices?

$$\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$ **[GATE 2005]**

(a) -1 and 1 (b) 1 and 6
(c) 2 and 5 (d) 4 and – 1

16. If I_n is an identity matrix of order n and k any scalar, then

(a) $\text{adj}(kI_n) = kI_n$ (b) $\text{adj}(kI_n) = k^n I_n$
(c) $\text{adj}(kI_n) = k^{n-1} I_n$ (d) None of these

17. If A is a symmetric matrix, then
(a) adj (A) is a non-symmetric matrix
(b) adj (A) is a symmetric matrix
(c) adj (A) does not exist
(d) None of the above

18. Let I_n be an identity matrix of order n, then

(a) $\text{adj}(I_n) = I_n$ (b) $\text{adj}(I_n) = 0$
(c) $\text{adj}(I_n) = nI_n$ (d) None of these

19. Every skew-symmetric matrix of odd order is
(a) singular (b) non-singular
(c) identity (d) None of these

20. The following vectors $\left(\frac{1}{4}, 0, -\frac{1}{4}\right), \left(\frac{1}{3}, -\frac{1}{3}, 0\right)$ and $\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ are

(a) linearly independent (b) linearly dependent
(c) constant (d) None of these

21. The following vectors (1, 9, 9, 8), (2, 0, 0, 8) and (2, 0, 0, 3) are
(a) linearly dependent (b) constant
(c) linearly independent (d) None of these

22. The following vectors $(-4, 2)$, $(9, 1)$ and (5, 3) are
(a) linearly dependent (b) linearly independent
(c) constant (d) None of these

23. Let $m \equiv$ rank of matrix A and $n \equiv$ number of linearly independent column vectors of matrix A, then
(a) $m < n$ (b) $m > n$
(c) $m \leq n$ (d) None of these

24. Let A and B are two equivalent matrices, then
(a) rank A = rank B (b) rank $A \neq$ rank B
(c) rank $A >$ rank B (d) None of these

25. The characteristic roots for the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are

(a) 5, 1, 1 (b) 5, 2, 2
(c) 0, 0, 0 (d) None of these

26. The value of the determinant $\begin{vmatrix} 2 & 2 & 2 \\ -5 & -3 & -3 \\ 0 & -1 & -1 \end{vmatrix}$ is

(a) zero (b) 2
(c) 9 (d) None of these

27. The value of the determinant $\begin{vmatrix} a & a' & a'' \\ b & b' & b'' \\ 3a & 3a' & 3a'' \end{vmatrix}$ is

(a) $a' b' a''$ (b) $3a' b'$
(c) 0 (d) None of these

28. The following have the solution $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \end{bmatrix}$

(a) $x = 1, y = 2$ (b) $x = y = 1$
(c) $x = y = 2$ (d) None of these

29. Let $A = [a_{ij}]$ be a matrix : $a_{ij} = k \neq 0$, for every i, j, then rank (A) is
(a) zero (b) one
(c) order of the matrix (d) None of these

30. Let A and B are two matrices, then
(a) rank $(AB) = $ rank $(B^T A^T)$
(b) rank $(AB) = $ rank $(A^T B^T)$
(c) rank $(AB) \neq $ rank $(AB)^T$
(d) None of the above

31. Let rank $(AB) = 5$, then
(a) rank $A \leq 5$ and rank $B \geq 5$
(b) rank $A \leq 5$ and rank $B \leq 5$
(c) rank $A \geq 5$ and rank $B \geq 5$
(d) None of the above

32. If A is any matrix, then
(a) AA^T are Skew-symmetric
(b) AA^T are symmetric
(c) AA^T are identity
(d) None of the above

33. If A and B are symmetric matrices, then AB are symmetric iff
(a) $AB \neq BA$ (b) $AB = BA$
(c) $AB > BA$ (d) $AB < BA$

34. If A and B symmetric, then
(a) $A + B$ is symmetric
(b) $A + B$ is Skew-symmetric
(c) $A + B$ is identity
(d) None of the above

35. If A is square matrix, then
(a) $A + A^T$ is symmetric and $A - A^T$ is Skew-symmetric
(b) $A + A^T$ and $A - A^T$ are Skew-symmetric
(c) $A - A^T$ and $A + A^T$ are symmetric
(d) None of the above

36. If A and B are Skew-symmetric, then
(a) $A + B$ is Skew-symmetric
(b) $A + B$ is symmetric
(c) $A + B$ is zero
(d) None of the above

37. If A is a symmetric matrix of rank-r, then A can reduce to
(a) diagonal matrix with exactly r, non-zero elements
(b) diagonal matrix with $k < r$, non-zero elements
(c) identity matrix
(d) None of the above

38. If A is a matrix that can be reduced to diagonal matrix, then
(a) $|A| = 0$ (b) $|A| \neq 0$
(c) $|A| > 0$ (d) None of these

39. Given $AX = 0$, where $A = [a_{ij}]_{n \times n}$ and $X = (x_1, \ldots, x_n)$ when rank $A = n$, then
(a) $x_1 = x_2 = \ldots = x_n = 0$ (trivial solution)
(b) $x_1 = x_2 = \ldots = x_n \neq 0$
(c) $x_1 \neq x_2 \neq x_3 \neq \ldots$
(d) None of the above

40. Let $A = [a_{ij}]_{m \times n}$ and $\tilde{A} = [a_{ij}, b_i]_{m \times (n+1)}$ for $AX = \overline{b}$, then
(a) A is coefficient matrix and $\tilde{A}$ is augmented matrix
(b) $\tilde{A}$ is coefficient matrix and A is augmented matrix
(c) A and $\tilde{A}$ are coefficient matrix
(d) None of the above

41. If rank of matrix A is 5 and nullity of A is 3, then the order of the matrix is
(a) 5 (b) 3
(c) 8 (d) zero

42. If Skew-Hermitian matrix A is real, then A is
(a) Skew-symmetric (b) symmetric
(c) Hermitian (d) None of these

43. If unitary matrix A is real, then A is
(a) symmetric matrix (b) Skew-symmetric matrix
(c) Hermitian matrix (d) orthogonal matrix

44. Every Hermitian matrix can be written as $A + iB$, where A and B are real then
(a) A is symmetric, B is Skew-symmetric
(b) A is hermitian, B is Skew-hermitian
(c) A is Skew-symmetric, B is symmetric
(d) A and B are both Skew-symmetric

45. Let $A \neq 0$ and B, C are matrices such that $AB = AC$, then
(a) $B = C$ (b) $B \neq C$
(c) $B \neq A$ (d) $C \neq A$

46. If A is orthogonal, then
(a) A^T and A^{-1} are both orthogonal
(b) A^T is orthogonal, but A^{-1} is not
(c) A^{-1} is orthogonal, but A^T is not
(d) None of the above

47. Given,
$$-x + y = 0$$
$$-x - 2y + 3z = 0$$
$$2x + y - 3z = 0$$
Then,
(a) $x = y = z$ (b) $x = y, z = 0$
(c) $x = z, y = 0$ (d) $y = z, x = 0$

48. The eigen values of triangular matrix are
(a) elements of leading diagonal
(b) elements of first row
(c) elements of first column
(d) None of the above

49. If λ is an eigen value of symmetric matrix, then
(a) λ is pure imaginary (b) λ is complex
(c) λ is real (d) None of these

50. Applying elementary transform to a matrix, its rank
(a) increases (b) decreases
(c) does not change (d) None of these

51. If λ_1, λ_2 and λ_3 are eigen values of matrix A, then A^3 has the eigen values
(a) $\lambda_1^2, \lambda_2^2, \lambda_3^2$ (b) $\lambda_1, \lambda_2, \lambda_3$
(c) $\lambda_1^3, \lambda_2^3, \lambda_3^3$ (d) $\lambda_1 = \lambda_2 = \lambda_3$

52. Match the following and select the correct answer using the codes given below the lists.

List I	List II
(A) If the matrix A is idempotent, then $(I-A)^n =$	(p) $2^{n-1}(I-A)$
(B) If the matrix A is involutory then $(I-A)^n =$	(q) $I - nA$
(C) If A is nilpotent matrix of order 2, then $(I-A)^n =$	(r) A
(D) If the matrix A is orthogonal, then $(A^T)^{-1} =$	(s) $I - A$

Codes

	A	B	C	D
(a)	s	p	q	r
(b)	p	q	s	r
(c)	s	r	q	p
(d)	r	p	s	q

53. The system of equations

$$x + y + z = 6$$
$$x + 4y + 6z = 20$$
$$x + 4y + \lambda z = \mu$$

has no solution for values of λ and μ given by **[EC GATE 2011]**
(a) $\lambda = 6, \mu = 20$ (b) $\lambda = 6, \mu \neq 20$
(c) $\lambda \neq 6, \mu = 20$ (d) $\lambda \neq 6, \mu \neq 20$

54. The two vectors [1, 1, 1] and $[1, a, a^2]$, where $a = \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$, are **[EE GATE 2011]**
(a) orthonormal (b) orthogonal
(c) parallel (d) collinear

55. The matrix $[A] = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$ is decomposed into a product of a lower triangular matrix [L] and an upper triangular matrix [U]. The properly decomposed [L] and [U] matrices, respectively, are **[EE GATE 2011]**
(a) $\begin{bmatrix} 1 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$
(b) $\begin{bmatrix} 2 & 0 \\ 4 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$
(d) $\begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix}$

56. For the set of equations,

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$
$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

which of the following statements is true? **[EE GATE 2010]**
(a) Only the trivial solution $x_1 = x_2 = x_3 = x_4 = 0$ exists
(b) There are no solutions
(c) A unique non-trivial solution exists
(d) Multiple non-trivial solutions exist

57. The eigen values of a (2×2) matrix X are -2 and -3. The eigen values of the matrix $(X+I)^{-1}(X+5I)$ are **[IE GATE 2009]**
(a) $-3, -4$ (b) $-1, -2$ (c) $-1, -3$ (d) $-2, -4$

58. One of the eigen vectors of the matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$ is **[ME GATE 2010]**
(a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ (b) $\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$ (c) $\begin{Bmatrix} 4 \\ 1 \end{Bmatrix}$ (d) $\begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$

59. Consider the matrix $P = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$. The value of e^P is **[EC GATE 2008]**
(a) $\begin{bmatrix} 2e^{-2} - 3e^{-1} & e^{-1} - e^{-2} \\ 2e^{-2} - 2e^{-1} & 5e^{-2} - e^{-1} \end{bmatrix}$
(b) $\begin{bmatrix} e^{-1} + e^{-2} & 2e^{-2} - e^{-1} \\ 2e^{-1} - 4e^{2} & 3e^{-1} + 2e^{-2} \end{bmatrix}$
(c) $\begin{bmatrix} 5e^{-2} - e^{-1} & 3e^{-1} - e^{-2} \\ 2e^{-2} - 6e^{-1} & 4e^{-2} + e^{-1} \end{bmatrix}$
(d) $\begin{bmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{bmatrix}$

60. It is given that $X_1, X_2, \ldots, X_M$ are M non-zero, orthogonal vectors. The dimension of the vector space spanned by the $2M$, vectors $X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M$ is **[EC GATE 2007]**

(a) $2M$

(b) $M+1$

(c) M

(d) dependent on the choice of $X_1, X_2, \ldots, X_M$

61. The linear operation $L(x)$ is defined by the cross product $L(x) = bX_k$, where $b = [0 \ 1 \ 0]^T$ and $x = [x_1 \ x_2 \ x_3]^T$ are three-dimensional vectors. The 3×3 matrix M of this operation satisfies

$$L(x) = M\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then, the eigen values of M are **[EE GATE 2007]**

(a) $0, +1, -1$ (b) $1, -1, 1$

(c) $i, -i, 1$ (d) $i, -i, 0$

62. The number of linearly independent eigen vectors of $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is **[ME GATE 2007]**

(a) 0 (b) 1 (c) 2 (d) infinite

63. $\bar{\bar{\mathbf{A}}}$ and $\bar{\bar{\mathbf{B}}}$ are two 3×3 matrices such that

$$\bar{\bar{\mathbf{A}}} = \begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}, \bar{\bar{\mathbf{B}}} = \bar{\bar{\mathbf{0}}}$$

and $\bar{\bar{\mathbf{A}}}\ \bar{\bar{\mathbf{B}}} = \bar{\bar{\mathbf{0}}}$. Then, the rank of matrix $\bar{\bar{\mathbf{B}}}$ is **[CS GATE 2007]**

(a) $r = 2$ (b) $r < 3$

(c) $r \le 3$ (d) $r = 3$

64. If the following represents the equation of a line

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

then the line passes through the point **[CS GATE 2006]**

(a) (0, 0) (b) (3, 4)

(c) (4, 3) (d) (4, 4)

65. For a given 2×2 matrix A, it is observed that

$$A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and $$A = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Then, matrix A is **[IE GATE 2006]**

(a) $A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

(d) $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

66. For the matrix $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, one of the eigen values is equal to -2. Which of the following is an eigen vector? **[EE GATE 2005]**

(a) $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$

67. X and Y are non-zero square matrices of size $n\times n$. If $XY = 0_{n\times n}$, then **[IE GATE 2010]**

(a) $|X| = 0$ and $|Y| \ne 0$ (b) $|X| \ne 0$ and $|Y| = 0$

(c) $|X| = 0$ and $|Y| = 0$ (d) $|X| \ne 0$ and $|Y| \ne 0$

68. The matrix A is given by

$$A = \begin{bmatrix} 1 & 4 \\ a & 2 \end{bmatrix}$$

The eigen values of the matrix A are real and non-negative for the condition **[CS GATE 2005]**

(a) $-\frac{1}{16} \le a \le \frac{1}{16}$ (b) $-\frac{1}{2} \le a \le \frac{1}{2}$

(c) $-\frac{1}{2} \le a \le \frac{1}{16}$ (d) $-\frac{1}{16} \le a \le \frac{1}{2}$

69. For which value of x, the matrix given below becomes singular?

$$\begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

[ME GATE 2004]

(a) 4 (b) 6 (c) 8 (d) 12

70. The inverse of the matrix $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ **[CS GATE 2000]**

(a) does not exist (b) $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix}$ (d) $\begin{bmatrix} -0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$

Common Data/Linked Answer Questions

Statement for Linked Answer Questions 1 and 2

Consider the matrix $A = \begin{bmatrix} 1 & 2010 & 2050 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

1. The eigen values of matrix A are
(a) 0, 1, 2 (b) 1, 2, 3
(c) 0, 1, −1 (d) None of these

2. The number of linearly independent eigen vectors is
(a) zero (b) 1 (c) 4 (d) 3

Statement for Linked Answer Questions 3 and 4
Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{bmatrix}$$

3. The diagonal matrix congruent to symmetric matrix A is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

4. If P is non-singular matrix such that $P^{-1}AP$ is diagonal matrix, then P^t is

(a) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 0 \\ 7 & 2 & 0 \end{bmatrix}$ (d) None of these

Statement for Linked Answer Questions 5 and 6
Consider the system of non-homogeneous linear equations

$$3x - y - 2z = 2$$
$$2x - z = -1$$
$$3x - 5y = 3$$

5. The rank of augmented matrix is
(a) zero (b) 1
(c) 2 (d) 3

6. The system has
(a) unique solution
(b) infinitely many solutions
(c) no solution
(d) any two of the above

Statements for Linked Answer Questions 7 and 8
Caylay-Hamilton theorem states that a square matrix satisfies its own characteristic equation. Consider a matrix,

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$$

7. A satisfies the relation **[EE GATE 2007]**
(a) $A + 3I + 2A^{-1} = 0$ (b) $A^2 + 2A + 2I = 0$
(c) $(A + I)(A + 2I) = 0$ (d) $\exp(A) = 0$

8. A^9 equals to **[EE GATE 2007]**
(a) $511A + 510$ (b) $309A + 104$
(c) $154A + 155$ (d) $\exp(9A)$

Statement for Linked Answer Questions 9 to 10

$P = \begin{bmatrix} -10 \\ 1 \\ 3 \end{bmatrix}^T, Q = \begin{bmatrix} -2 \\ -5 \\ 9 \end{bmatrix}^T, R = \begin{bmatrix} 2 \\ -7 \\ 12 \end{bmatrix}^T$ are three vectors.

9. An orthogonal set of vectors having a span that contains P, Q, R is **[EE GATE 2006]**

(a) $\begin{bmatrix} -6 \\ -3 \\ -6 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ -11 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \\ -3 \end{bmatrix}$

(c) $\begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -2 \end{bmatrix} \begin{bmatrix} 3 \\ 9 \\ -4 \end{bmatrix}$ (d) $\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$

10. Which of the following vectors is linearly dependent upon the solution to the previous problem? **[EE GATE 2006]**

(a) $\begin{bmatrix} 8 \\ 9 \\ 3 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -17 \\ 30 \end{bmatrix}$ (c) $\begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$

Statement for Linked Answer Questions 11 and 12
A system of linear simultaneous equations is given as

$$Ax = b, \text{ where } A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

11. The rank of matrix A is **[IE GATE 2006]**

(a) 1 (b) 2
(c) 3 (d) 4

12. Which of the following statements is true? **[IE GATE 2006]**

(a) x is a null vector
(b) x is unique
(c) x does not exist
(d) x has infinitely many values

Common Data for Questions 13 and 14

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

13. The Echelon form of matrix A is

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 2 & 6 & 2 & -6 & 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & -2 & 1/3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 1 & 2 & -2 & 1/3 \\ 0 & 0 & 1 & -2 & 1/3 \end{bmatrix}$

(d) None of the above

14. Rank of matrix A is

(a) zero (b) 1 (c) 2 (d) 3

Common Data for Questions 15 and 16

Consider the system of equations

$$2x + 3y - z = 0$$
$$x - y - 2z = 0$$
$$3x + y + 3z = 0$$

15. Rank of coefficient matrix is

(a) zero (b) 1
(c) 2 (d) 3

16. The system has

(a) trivial solution
(b) non-trivial solutions
(c) no solutions
(d) None of the above

Common Data for Questions 17 and 18

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}_{100 \times 100}$$

17. Characteristic polynomial of A is

(a) $x^{100}(x-100)$ (b) $x^{99}(x-100)$
(c) $x^{100}(x-99)$ (d) $x^{99}(x-99)$

18. Minimal polynomial of A is

(a) $x^2 + 1$ (b) $1 - x^2$
(c) $x^2 - 1$ (d) $x^2 - 100x$

Common Data for Questions 19 and 20

An n-square matrix A is orthogonal, if $A'A = I$, where I is identity matrix.

19. Matrix A is

(a) singular (b) $A^{-1} = A^t$
(c) A^{-1} does not exist (d) $A^{-1} = A$

20. $|A|$ is

(a) $\pm\frac{1}{2}$ (b) $\pm\frac{4}{5}$
(c) ± 1 (d) zero

Answers with Solutions

Unit Exercise 1

1. (a) $4A + 3X = 5B \Rightarrow 3X = 5B - 4A$

$$\Rightarrow \quad X = \frac{1}{3}(5B - 4A)$$

2. (b) $B = (2A - 3B) - 2(A - 2B)$

3. (b) $2x + 1 = 5 \Rightarrow x = 2, \quad 8 + y = 0 \Rightarrow y = -8$

4. (b)
$$(A_\alpha)^2 = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
$$= \begin{bmatrix} \cos^2\alpha - \sin^2\alpha & \cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha - \cos\alpha\sin\alpha & -\sin^2\alpha + \cos^2\alpha \end{bmatrix}$$
$$= \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

5. (b, d) From characteristic equation $\lambda = 0$

So, if $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is eigen vector.

Then, $\quad AX = \lambda X$

$$\Rightarrow \quad \begin{bmatrix} \alpha z \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad az = 0$$

$\because a = 0$, we have $z = 0$. Hence, all vectors of the form $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$ are eigen vector so options (b) and (d) are correct.

6. (d) $\because$ A and B are real symmetric matrices of size $n \times n$. Therefore, AB is symmetric iff $AB = BA$.

Thus, $\quad (AB)' = AB$
$$= BA$$

7. (c) Given, matrix

$$A = \begin{bmatrix} 0 & 0 & -3 \\ 9 & 3 & 5 \\ 3 & 1 & 1 \end{bmatrix}$$

$\therefore \quad |A| = 0(3-5) + 0(15-9) - 3(9-9)$
$= 0 + 0 + 0 = 0$

$\therefore \quad \rho(A) < 3$

But the determinant of submatrix

$$\begin{bmatrix} 3 & 5 \\ 1 & 1 \end{bmatrix} = 3 - 5 = -2 \neq 0$$

$\therefore \quad \rho(A) = 2$

8. (c) In array row 1 is stored first, row 2 next. Row 3 next, and so on. Here, row 1 has one element. Row 2 has 2 elements. Row 3 has 3 elements, …… Row $(i-1)$ has $(i-1)$ elements. Hence, (i, j)th elements present in the array only after presence of

1 element of row 1

2 element of row 2,

…………………

$(i-1)$ element of row $(i-1)$ and first $(i-1)$ element of row i.

$\Rightarrow$ The index of jth element in the array is

$$1 + 2 + 3 + \ldots + (i-1) + j = \frac{i(i-1)}{2} + j$$

9. (a) Given, matrix

$$A = \begin{bmatrix} 1 & a & a^2 & \ldots & a^n \\ 1 & a & a^2 & \ldots & a^n \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & a & a^2 & \ldots & a^n \end{bmatrix}_{(n+1)\times(n+1)}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$
………………
………………
$R_n \to R_n - R_1$
$R_{n+1} \to R_{n+1} - R_1$

$$\sim \begin{bmatrix} 1 & a & a^2 & \ldots & a^n \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{(n+1)\times(n+1)}$$

which is in Echelon form and the number of non-zero rows is 1. Hence, $\rho(A) = 1$.

10. (b) Given, matrix

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Now, $\quad A + A' = I$

$$\Rightarrow \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad 2\cos\alpha = 1$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad \alpha = \frac{\pi}{3}$$

11. (c) A is singular, if $|A| = 0$

12. (c) **13.** (c) **14.** (b) **15.** (c) **16.** (a) **17.** (d)

18. (b) $\because ABCD = I \Rightarrow AB = D^{-1}C^{-1}$

$$\Rightarrow \quad (AB)^{-1} = (D^{-1}C^{-1})^{-1} = CD$$

$$\Rightarrow \quad B^{-1}A^{-1} = CD$$

$$\Rightarrow \quad B^{-1} = CDA$$

19. (c)

20. (a) $\because$ A and B are idempotent.

$$\Rightarrow \quad A^2 = A \quad \text{and} \quad B^2 = B$$

$$\Rightarrow \quad (AB)^2 = AB \cdot AB = A^2B^2 \text{ iff } AB = BA$$

$$\Rightarrow \quad (A-B)^2 = A^2B^2 = AB \text{ iff } AB = BA$$

21. (a) $\bar{A} = \begin{bmatrix} 1-i & -i \\ 2 & 1+i \end{bmatrix}$

$$\Rightarrow \quad (\bar{A})' = \begin{bmatrix} 1-i & 2 \\ -i & 1+i \end{bmatrix}$$

22. (a) A and B are idempotent.

If $A^2 = A$ and $B^2 = B$

$(A + B)$ is idempotent.

If $(A + B)^2 = A + B$

Here, $(A + B)^2 = A^2 + AB + BA + B^2$

$= A + AB + BA + B$

$\because$ $(A + B)$ is idempotent, it is possible only when

$AB = BA = 0$

23. (a) If every element in any row (or column) is multiplied by same factor, the determinent is multiplied by that factor. $\because$ Each column (or row) is multiplied by k and number of rows are n.

So, $|kA| = k^n|A|$

24. (a) diag $(1, \ldots, 1) = I$ and $I^2 = I$

25. (c) $\begin{bmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -13 \\ -5 \\ 2 \end{bmatrix}$

By Cramer's rule,

$$\frac{x}{\begin{vmatrix} -13 & -5 & 2 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 3 & 7 & 8 \\ -13 & -5 & 2 \\ -4 & 1 & -26 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -13 & -5 & 2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 7 & 8 \\ 2 & 0 & 9 \\ -4 & 1 & -26 \end{vmatrix}}$$

$x = -7,\ y = 0,\ z = 1$

26. (d) **27.** (a) **28.** (a)

29. (a) $\because$ For characteristic roots $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{vmatrix} 3+4i-\lambda & -5i \\ -7 & 6-2i-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (3+4i-\lambda)(6-2i-\lambda) - 35i = 0$$

$$\Rightarrow \quad \lambda^2 - 11\lambda + 18 = 0$$

$$\Rightarrow \quad \lambda = 2, 9$$

30. (d) Applying $R_1 \to aR_1$, $R_2 \to bR_2$ and $R_3 \to cR_3$, we have

$$\Delta = \frac{1}{abc}\begin{vmatrix} ab^2c & abc & ab+ac \\ a^2bc^2 & abc & bc+ab \\ a^2b^2c & abc & ac+bc \end{vmatrix}$$

$$= \frac{a^2b^2c^2}{abc}\begin{vmatrix} bc & 1 & ab+ac \\ ac & 1 & bc+ab \\ ab & 1 & ac+bc \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1$

$$\Delta = abc\begin{vmatrix} bc & 1 & ab+ac+bc \\ ac & 1 & ab+ac+bc \\ ab & 1 & ab+ac+bc \end{vmatrix}$$

$$= abc(ab+ac+bc)\begin{vmatrix} bc & 1 & 1 \\ ac & 1 & 1 \\ ab & 1 & 1 \end{vmatrix}$$

$= 0$ ($\because C_1$ and C_2 are identical)

31. (c) $\Delta = \begin{vmatrix} m & n & p \\ p & m & n \\ n & p & m \end{vmatrix}$

$$= m\begin{vmatrix} m & n \\ p & m \end{vmatrix} - n\begin{vmatrix} p & n \\ n & m \end{vmatrix} + p\begin{vmatrix} p & m \\ n & p \end{vmatrix}$$

$$= m(m^2 - np) - n(pm - n^2) + p(p^2 - mn)$$

$$= m^3 + n^3 + p^3 - 3\,mnp$$

32. (b) **33.** (c)

34. (b) $\because A = \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix}$

$$\Rightarrow \quad |A| \neq 0$$

So, rank of $A = 3$

35. (a) **36.** (c) **37.** (a) **38.** (c) **39.** (a)
40. (a) **41.** (b) **42.** (a) **43.** (a) **44.** (a)
45. (b) **46.** (a) **47.** (a) **48.** (a) **49.** (b)
50. (a)

51. (a) $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$

Let $B = (-2A + 3I)$

$$= -2\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 \\ -8 & -3 \end{bmatrix}$$

Now, $|B - \lambda I| = \begin{bmatrix} 1-\lambda & -4 \\ -8 & -3-\lambda \end{bmatrix} = 0$

$\Rightarrow \quad (1-\lambda)(-3-\lambda) - 32 = 0$

$\Rightarrow \quad -3 + 2\lambda + \lambda^2 - 32 = 0$

$\Rightarrow \quad \lambda^2 + 2\lambda - 35 = 0$

$\Rightarrow \quad \lambda^2 + 7\lambda - 5\lambda - 35 = 0$

$\Rightarrow \quad \lambda(\lambda + 7) - 5(\lambda + 7) = 0$

or $\quad (\lambda + 7)(\lambda - 5) = 0$

$\Rightarrow \quad \lambda = -7, 5$

Thus, eigen values of $(-2A + 3I)$ are -7 and 5.

52. (a) Given, $x = [x_1 \, x_2 \ldots x_n]^T$

Since, x is an n-tuple non-zero vector, that is, x is a non-singular matrix of order n, so its rank should be n.

i.e., $\quad I(x) = n$

The vector (x^T) also have rank (n) because transpose of any matrix does not altered its rank.

Then, matrix$[v = xT']$ must have the rank n, *i.e.*, $I(v) = n$, because resultant of the multiplication of two same rank matrices also has the same rank as the rank of multiplicative matrix.

53. (b) Given, $a = (a_{ij})_{n \times n}$

$A^2 = I$, yn-dimensional vector.

As, $\quad A^2 = I \Rightarrow A \cdot A = I$

$\Rightarrow \quad |A \cdot A| = |I|$

$\Rightarrow \quad |A||A| = 1$

$\Rightarrow \quad |A|^2 = 1$

$\Rightarrow \quad |A| \neq 0$

$\Rightarrow \quad A$ is non-singular.

Rank of $A = n$

So, $Ax = y$ has unique solution.

54. (c) By Echelon form; we know that the rank of any matrix is equal to the number of non-zero rows.

Here, the given matrix is of order 3×4, *i.e.*, only three rows. So, the highest possible rank should be 3.

55. (d) The necessary condition to diagonalize a matrix A is that $|A| \neq 0$, *i.e.*, A must be non-singular.

(As $|A| = \lambda_1 \cdot \lambda_2 \ldots \lambda_n$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A).

56. (d) Let $X = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$

We take, $Y = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$

Then, $\quad XY = I = YX$

$\Rightarrow \quad Y = X^{-1}$

57. (a) $|x| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 \\ -1 & 4 & 7 & 4 & 0 \\ -5 & -6 & 2 & 1 & 1 \end{vmatrix}$

$$= 1\begin{vmatrix} 2 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 \\ 4 & 7 & 4 & 0 \\ -6 & 2 & 1 & 1 \end{vmatrix} + 0$$

$$= 2\begin{vmatrix} 3 & 0 & 0 \\ 7 & 4 & 0 \\ 2 & 1 & 1 \end{vmatrix} + 0$$

$$= 2 \cdot 3 \cdot \begin{vmatrix} 4 & 0 \\ 1 & 1 \end{vmatrix} + 0 = 6(4) = 24$$

Thus, $|x| = 24$

Unit Exercise 2

1. (c) **2.** (c) **3.** (a)

4. (b) $\begin{vmatrix} 6 & -8 & 1 & 1 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & 8 \\ 0 & 0 & 0 & -1 \end{vmatrix} = 6\begin{vmatrix} 2 & 4 & 6 \\ 0 & 4 & 8 \\ 0 & 0 & -1 \end{vmatrix}$

$$= 6 \times 2\begin{vmatrix} 4 & 8 \\ 0 & -1 \end{vmatrix} = 6 \times 2 \times -4 = -48$$

5. (c) $A = a_{ij} = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} \\ a_{21} & a_{22} & \ldots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} \end{bmatrix}$

and $I = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 \end{bmatrix}$

$$A\,I_{12} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} \\ a_{21} & a_{22} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \phi \end{bmatrix}$$

$$= \begin{bmatrix} a_{12} & a_{11} & \cdots & a_{1j} \\ a_{22} & a_{21} & \cdots & a_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i2} & a_{i1} & \cdots & a_{ij} \end{bmatrix}$$

6. (a) $\because R_4 = 3R_1$ so, rank < 4

and $\begin{vmatrix} 1 & 4 & 8 \\ 0 & 0 & 3 \\ 4 & 2 & 3 \end{vmatrix} = 48 \neq 0$

So, rank = 3

7. (b)

8. (b) By theorem, Rank $(AB) \leq$ Min (rank A, rank B)

$\therefore$ Rank (PQ) = Min (Rank P, rank Q)

Since, P is a $m \times n$ matrix with rank n, $n \leq m$ and Q is $n \times P$ matrix with rank P, $P \leq n$.

$\therefore$ Rank (PQ) = Min $(n, P) = P$

9. (d) If $\lambda = 0$ is an eigen value of P, then eigen value of Q is $\frac{1}{\lambda}$.

i.e., ∞, but this is not possible.

Hence, neither P nor Q.

10. (a) **11.** (c)

12. (a) From $|A - \lambda I|\, X = 0$

13. (b) We can write the following linear equations

$$2x + y - 4z = \alpha$$
$$4x + 3y - 12z = 5$$
$$x + 2y - 8z = 7$$

For infinitely solution $D = 0$

$$\begin{vmatrix} 2 & 1 & -4 \\ 4 & 3 & -12 \\ 1 & 2 & -8 \end{vmatrix} = 0$$

$\because$ 2nd and 3rd columns are linearly dependent. Now, we find the values of x, y and z.

So, for x, $D = \begin{vmatrix} a & 1 & -4 \\ 5 & 3 & -12 \\ 7 & 2 & -8 \end{vmatrix} = 0$

$\Rightarrow \quad \alpha = \frac{1}{5}$

$\because$ 2nd and 3rd columns are linearly dependent.

For y, $\quad D = \begin{vmatrix} 2 & \alpha & -4 \\ 4 & 5 & -12 \\ 1 & 7 & -8 \end{vmatrix} = 0$

$\Rightarrow \quad \alpha = \frac{1}{5}$

For z, $\quad D = \begin{vmatrix} 2 & 1 & \alpha \\ 4 & 3 & 5 \\ 1 & 2 & 7 \end{vmatrix} = 0 \Rightarrow \alpha = \frac{1}{5}$

Hence, from above equations α has only one value for infinitely solution.

14. (a) Given matrix is $M \times N$ has a rows and b columns.

Then maximum number of non-zero entries such that no two are on the same row and column is less than the sum of number of rows and columns

Sum = $a + b$

$\therefore$ Number of zero entries $\leq a + b$.

15. (b) $\lambda I - A = 0$

16. (c) (kI_n) adj $(kI_n) = |kI_n|\, I_n$

$\Rightarrow \quad (kI)_n$ adj $(kI_n) = k^n I_n$

$\Rightarrow \quad$ adj $(kI_n) = k^{n-1} I_n$

$\because \quad$ (adj $A)^T$ = adj A^T

17. (b) $\Rightarrow A$ is symmetric, if $A^T = A$

(adj $A)^T$ = adj (A)

18. (a) $\therefore$ adj (A) is symmetric.

(adj $A)\, A = |A|\, I_n$

$\Rightarrow \quad$ (adj $I_n)\, I_n = |I_n|\, I_n = I_n$

$\Rightarrow \quad$ adj $(I_n) = I_n$

19. (a)

20. (a) $\begin{vmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ \frac{1}{3} & -\frac{1}{3} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{3} \end{vmatrix} \neq 0$

$|A| \neq 0 \Leftrightarrow$ rows (columns) are independent.

21. (c) **22.** (a) **23.** (c)

24. (a) Transformation does not alter the rank of matrix.

25. (a) **26.** (a) **27.** (c) **28.** (a)

29. (b) Matrix with each element as unity is one.

$|k| = k \neq 0$

$\begin{vmatrix} k & k \\ k & k \end{vmatrix} = k \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = k.0 = 0$

30. (a) $\because$ Transformation does not alter the rank of matrix

$\therefore$ Rank (AB) = rank $(AB)^T$

= rank $(B^T A^T)$

31. (b) If rank $(AB) = k$

$\Rightarrow \quad$ rank $A \leq k$,

rank $B \leq k$

32. (b) A and B are symmetric iff

$\Rightarrow \quad A^T = A,\ B^T = B$

33. (b) $\therefore (AB)^T = B^T A^T = BA$

$= AB$ (iff A and B commute)

$(BA)^T = A^T B^T = AB$

$= BA$ (iff A and B commute)

34. (a) A and B are symmetric.

$$\Rightarrow \quad A = A^T, B = B^T$$

$$(A + B)^T = A^T + B^T = (A + B)$$

35. (a) $\Rightarrow$ $(A + B)$ is symmetric.

$$(A + A^T)^T = A^T + A = (A + A^T)$$

$$(A - A^T)^T = A^T - A = -(A - A^T)$$

36. (a) A and B are skew-symmetric.

$$\Rightarrow \quad A = -A^T, B = -B^T$$

$$(A + B)^T = A^T + B^T = -A - B$$

$$= -(A + B)$$

37. (a) **38.** (b) **39.** (a) **40.** (a) **41.** (c)
42. (c) **43.** (a) **44.** (b) **45.** (b) **46.** (c)
47. (b) **48.** (a) **49.** (c) **50.** (c) **51.** (c)

52. (a)

(A) $\because$ A is idempotent.

$$\Rightarrow \quad A^2 = A$$

$$\Rightarrow \quad A^3 = A \cdot A^2 = A \cdot A = A^2 = A$$

$$\Rightarrow \quad A^4 = A \cdot A^3 = A \cdot A = A^2 = A$$

$$\Rightarrow \quad A^n = A$$

$$\Rightarrow \quad (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$$

$$= I + (-{}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots)$$

$$= I + [({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots) - {}^nC_0]\, A$$

$$= I - A$$

(B) $\because$ A is involutory.

$$\Rightarrow \quad A^2 = I$$

$$\Rightarrow \quad A^3 = A^4 = A^5 = \dots = I$$

$$\Rightarrow \quad (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$$

$$= {}^nC_0 I - {}^nC_1 A + {}^nC_2 I - {}^nC_3 A + \dots$$

$$= ({}^nC_0 + {}^nC_2 + {}^nC_4 + \dots)\, I - ({}^nC_1 + {}^nC_3 + \dots)\, A$$

$$= 2^{n-1} (I - A)$$

(C) $\because$ A is nilpotent.

$$\therefore \quad A^2 = A^3 = \dots = A^n = 0$$

$$\Rightarrow \quad (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$$

$$= I - nA$$

(D) $\because$ A is orthogonal.

$$\therefore \quad AA' = I$$

$$\Rightarrow \quad (A')^{-1} = A$$

53. (b) Given, equations are $x + y + z = 6$, $x + 4y + 6z = 20$ and $x + 4y + \lambda z = \mu$

If $\lambda = 6$ and $\mu = 20$, then $x + 4y + 6z = 20$

$$x + 4y + 6z = 20$$

we have infinite solution.

If $\lambda = 6$ and $\mu \neq 20$, then $x + 4y + 6z = 20$

$$x + 4y + 6z = 20$$

We have no solution.

If $\quad \lambda \neq 6 \quad$ and $\quad \mu = 20$

$$x + 4y + 6z = 20$$

$$x + 4y + \lambda z = 20$$

will have solution.

If $\quad \lambda \neq 6 \quad$ and $\quad \mu \neq 20$

will also have solution.

54. (b) These two vectors are orthogonal because the dot product of these two vectors is zero.

55. (d) $|A| = |L|\,|U|$

$$= \begin{bmatrix} 2 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0.5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0 & 1+0 \\ 4+0 & 2-3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

56. (d) Given, system of equations

$$x_1 + 2x_2 + x_3 + 4x_4 = 2$$

$$3x_1 + 6x_2 + 3x_3 + 12x_4 = 6$$

which is the non-homogeneous system of equations.

The augmented matrix is

$$[A : B] = \begin{bmatrix} 1 & 2 & 1 & 4 & \vdots & 2 \\ 3 & 6 & 3 & 12 & \vdots & 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - 3R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 1 & 4 & \vdots & 2 \\ 0 & 0 & 0 & 0 & \vdots & 0 \end{bmatrix}$$

Here, $\quad$ Rank of $[A \vdots B] = 1$

and $\quad$ Rank of $A = 1$

Since, $\quad f[A \vdots B] = f[A]$

So, the given system of equations is consistent.

But, the rank of $A <$ the number of unknowns.

So, the system of non-homogeneous equations will have infinitely many solutions.

In other words, multiple non-trivial solutions exist.

57. (c) $\lambda_1 = -2, \lambda_2 = -3$

$$(x + I)^{-} (X + 5I) = ?$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} -9 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 1 \end{bmatrix}$$

58. (a) $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$A - \lambda I = 0 \qquad [\lambda = \text{Eigen value}]$$

$$\therefore \quad \begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow \quad (2 - \lambda)(3 - \lambda) - 2 = 0$$

or $\quad \lambda^2 - 5\lambda + 4 = 0$

$$\Rightarrow \quad \lambda = 1, 4$$

For eigen value, $\quad \lambda = 1$

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

Also, $\quad [A - \lambda I]x = 0$

where, x is the eigen vector.

$$\Rightarrow \quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or $\quad x_1 + 2x_2 = 0$

$$\Rightarrow \quad x_2 = -\frac{x_1}{2}$$

Only option (a) $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$ satisfies the condition.

Hence, eigen vector is $\begin{Bmatrix} 2 \\ -1 \end{Bmatrix}$.

59. (d)
$$P = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$P^2 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}$$

$$e^P = I + \frac{P}{1!} + \frac{P^2}{2!} + \frac{P^3}{3!} + \ldots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{1}{1!}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix} + \ldots$$

$$= \begin{bmatrix} 1 - \frac{2}{2!} + \frac{6}{3!} \ldots & 1 - \frac{3}{2!} + \frac{7}{3!} \ldots \\ -\frac{2}{1!} + \frac{6}{2!} - \frac{14}{3!} \ldots & 1 - \frac{3}{1!} + \frac{7}{2!} - \frac{15}{3!} + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-1} - e^{-2} & e^{-1} - e^{-2} \\ -2e^{-1} + 2e^{-2} & -e^{-1} + 2e^{-2} \end{bmatrix}$$

60. (a) $\because X_1, X_2, X_3, \ldots, X_M$ are M non-zero orthogonal vectors.

$\therefore$ Dim $(M) = M$

For $\quad X_1, -X_M, -X_1, \ldots, -X_M$

the extensions of above given vectors are nothing but rearrangement of above M vectors.

So, dimension of vector space spanned $= M$

For $X_1, X_2, \ldots, X_M, -X_1, -X_2, \ldots, -X_M$

Dim $(2M) = 2M$

$\therefore$ Vector space spanned by $2M = 2M$

61. (d) Since, $L(x) = M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then, the eigen values of M are $i, -i, 0$.

62. (b) $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

$A - \lambda I = 0 \quad$ [where λ is the eigen value]

$$\Rightarrow \quad \begin{bmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

or $\quad (2-\lambda)^2 = 0$

$\Rightarrow \quad \lambda = 2, 2$

So, only one eigen vector.

63. (b) Given, $A = \begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 1 \\ 0 & 4 & 4 \end{bmatrix}$

$$B = 0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$AB = 0$

Problem is to find rank of B.

Since, rank of a matrix is of the order of non-zero minor such that each minor of higher order is zero. Since, each entry of B is zero, so each minor of order 1 is zero and so rank of $B = 0 \Rightarrow$ rank of $B < 3$

Alternate method

Since, B is null matrix so rank of $B = 0 \Rightarrow$ rank of $B < 3$.

64. (b) Equation of a line is

$$\begin{vmatrix} x & 2 & 4 \\ y & 8 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad \ldots\text{(i)}$$

$\Rightarrow \quad x(8) + 2(-y) + 4(y - 8) = 0$

$\Rightarrow \quad 8x + 2y - 32 = 0$

$\Rightarrow \quad 4x + y - 16 = 0 \qquad \ldots$(ii)

We take point (3, 4).

Then, $4x + y - 16 = 4 \times 3 + 4 - 16$

$= 12 + 4 - 16 = 0$

$\Rightarrow$ Point (3, 4) satisfies Eq. (ii) and line (i). So, line (i) passes through (3, 4).

65. (c) We consider

$$A = \left(\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\right) \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Now, $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

66. (d) $P = \begin{bmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Also, eigen value $\lambda = -2$

Let $X_1 = [x_1 x_2 x_3]^T$ be eigen vector of the corresponding eigen value $\lambda = -2$, then the characteristic matrix

$= A - \lambda I$

$= A + 2I$

$$= \begin{bmatrix} 3+2 & -2 & 2 \\ 0 & -2+2 & 1 \\ 0 & 0 & 1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad [\text{Apply } R_3 \to R_3 - 3R_2]$$

$$\sim \begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Here, the rank of $P = 2$

Also, the rank of P is less than the number of unknowns, then the system will have infinitely many solutions, then the matrix form of given system is

$$AX_1 = 0$$

$$\begin{bmatrix} 5 & -2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$5x_1 - 2x_2 + 2x_3 = 0$$

$$x_3 = 0$$

Let $x_2 = k$

Then, $5x_1 = 2k$

$$x_1 = \frac{2k}{5}$$

Then, the corresponding eigen vector X_1 is

$$x_1 : x_2 : x_3 = \frac{2k}{5} : k : 0$$

$$x_1 : x_2 : x_3 = \frac{2}{5} : 1 : 0$$

$$x_1 : x_2 : x_3 = 2 : 5 : 0$$

$$\Rightarrow \quad X_1 = [2, 5, 0]^T$$

67. (a, b, c) Given, $X \neq 0, Y \neq 0$

$$X = |X_{ij}|_{n \times n}, Y = [Y_{ij}]_{n \times n}$$

and $XY = O_{n \times n}$

We know $|XY| = |0|$

$\Rightarrow \quad |X||Y| = 0$

We know that determinant of a non-zero square matrix may be zero.

So, $|X||Y| = 0$

Either $|X| = 0, |Y| \neq 0$

or $|X| \neq 0, |Y| = 0$

or $|X| = 0, |Y| = 0$

68. (d) $A = \begin{bmatrix} 1 & 4 \\ a & 2 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 4 \\ a & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)(2-\lambda) - 4a = 0$$

$$\Rightarrow \quad 2 + \lambda^2 - 3\lambda - 4a = 0$$

$$\Rightarrow \quad \lambda^2 - 3\lambda + (2 - 4a) = 0$$

$$\Rightarrow \quad \lambda = \frac{3 \pm \sqrt{9 - 4(2-4a)}}{2} = \frac{3 \pm \sqrt{1+16a}}{2}$$

As $\lambda \geq 0 \Rightarrow 1 + 16a \geq 0 \Rightarrow a \geq -\frac{1}{16}$

Also, $\lambda \geq 0 \Rightarrow \frac{3 - \sqrt{1+16a}}{2} \geq 0$

$$\Rightarrow \quad 3 \geq \sqrt{1+16a} \Rightarrow 9 \geq 1 + 16a$$

$$\Rightarrow \quad 8 \geq 16a \Rightarrow \frac{1}{2} \geq a$$

Thus, $-\frac{1}{16} \leq a \leq \frac{1}{2}$

69. (a) For singular matrix

$$A = \begin{bmatrix} 8 & x & 0 \\ 4 & 0 & 2 \\ 12 & 6 & 0 \end{bmatrix}$$

Since, for signular matrix

$$|A| = 0$$

$$\Rightarrow \quad 8[0 - 12] - x[0 - 2 \times 12] = 0$$

$$\Rightarrow \quad -96 + 24x = 0$$

$$\Rightarrow \quad x = 4$$

70. (c) Given, $A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

Problem is to find A^{-1}.

Here, $|A| = 1(-1) - (-1)(-1)$

$= -1 - 1 = -2 \neq 0$

$$\text{adj}(A) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

So, $$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}{-2}$$

$$= \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & -0.5 \end{bmatrix}$$

Common Data/Linked Answer Questions

1. (c) The characteristic polynomial of the matrix is

$$\text{ch}(x) = x(x-1)(x+1)$$

$\therefore$ Eigen values are 0, 1, –1.

2. (c) Construct the matrix

$$B_0 = A - 0.I = A = \begin{bmatrix} 1 & 2010 & 2050 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(B_0) = 2$

$\therefore$ Number of linearly independent vectors corresponding to $\lambda = 0$ is $3 - 2 = 1$.

$$B_1 = A - I = \begin{bmatrix} 0 & 2010 & 2050 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rho(B_1) = 2$$

$\therefore$ Number of linearly independent eigen vectors corresponding to $\lambda = 1$ is $3 - 2 = 1$.

$$B_{-1} = A + I = \begin{bmatrix} 2 & 2010 & 2050 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(B_{-1}) = 1$

$\therefore$ Number of linearly independent vectors corresponding to $\lambda = -1$ is $3 - 1 = 2$.

$\therefore$ Total required number $= 1 + 1 + 2 = 4$

3. (a) The block matrix

$$(A, I) = \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 5 & -4 & 0 & 1 & 0 \\ -3 & -4 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right] \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 3R_1 \end{pmatrix}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 2 & -1 & 3 & 0 & 1 \end{array}\right] \begin{pmatrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + 3C_1 \end{pmatrix}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right] (R_3 \to R_3 - 2R_2)$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -2 & 1 \end{array}\right] (C_3 \to C_3 - 2C_2)$$

Hence, the required diagonal matrix is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$

4. (b) From solution 3, $P' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 7 & -2 & 1 \end{bmatrix}$

5. (d) The given system of equations can be written in matrix form as

$$AX = B$$

where $A = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

and $B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$

The augmented matrix is

$$[A : B] = \left[\begin{array}{ccc:c} 3 & -1 & -2 & 2 \\ 2 & 0 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & -1 & 3 \\ 2 & 0 & -1 & -1 \\ 3 & -5 & 0 & 3 \end{array}\right] (R_1 \to R_1 - R_2)$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & -1 & 3 \\ 0 & 2 & 1 & -7 \\ 0 & -2 & 3 & -6 \end{array}\right] \begin{pmatrix} R_2 \to R_2 - 2R_2 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & -1 & 3 \\ 0 & 2 & 1 & -7 \\ 0 & 0 & 4 & -13 \end{array}\right] R_3 \to R_3 + R_2$$

which is in Echelon form, therefore

$$\rho[A : B] = 3$$

Also, $\rho(A) = 3$

$\therefore \quad \rho(A) = \rho[A : B] = 3 =$ Number of variables

6. (a) From solution 5, the system has unique solution.

7. (a) Given, $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} (-3-\lambda) & 2 \\ -1 & (0-\lambda) \end{vmatrix} = 0$$

$$\Rightarrow \quad (-3-\lambda)(-\lambda) + 2 = 0$$

$$\Rightarrow \quad 3\lambda + \lambda^2 + 2 = 0$$

$$\Rightarrow \quad \lambda^2 + 3\lambda + 2 = 0 \qquad \ldots\text{(i)}$$

Now, by Cayley-Hamilton theorem, "Every square matrix satisfies its characteristic equation.

Then, putting $\lambda = A$ in Eq. (i)

$$A^2 + 3A + 2 = 0 \qquad \ldots\text{(ii)}$$

Operating (A^{-1}) on both sides,

$$(A^{-1}A)A + 3(A^{-1}A) + 2A^{-1} = 0 \quad [\because AA^{-1} = I, AI = A]$$

$$I \cdot A + 3I + 2A^{-1} = 0$$

$$A + 3I + 2A^{-1} = 0$$

8. (a) From Eq. (ii)

$$A^2 + 3A + 2 = 0$$

$$\Rightarrow \quad A^2 = -3A - 2$$

$$\Rightarrow \quad A^3 = -3A^2 - 2A$$

$$= -3(-3A - 2) - 2A$$

$$= 9A - 2A + 6$$

$$A^3 = 7A + 6$$

$$\Rightarrow \quad A^4 = 7A^2 + 6A = 7(-3A - 2) + 6A$$

$$= -15A - 14$$

$$\Rightarrow \quad A^5 = 7A^3 + 6A^2$$

$$= 7(7A + 6) + 6(-3A - 2)$$

$$= 49A + 42 - 18A - 12$$

$$A^5 = 31A + 30$$

$$\Rightarrow \quad A^6 = 31A^2 + 30A$$

$$\Rightarrow \quad A^7 = 31A^3 + 30A^2$$

$$\Rightarrow \quad A^8 = 31A^4 + 30A^3$$

$$\Rightarrow \quad A^9 = 31A^5 + 30A^4$$

$$= 31(31A + 30) + 30(-15A - 14)$$

$$= 961A + 930 - 450A - 420$$

$$A^9 = 511A + 510$$

9. (d) The orthogonal set of vectors are

$$\begin{bmatrix} 4 \\ 3 \\ 11 \end{bmatrix} \begin{bmatrix} 1 \\ 31 \\ 3 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ 4 \end{bmatrix}$$

10. (d) Hence, the vector is linearly dependent upon the solution is

$$\begin{bmatrix} 13 \\ 2 \\ -3 \end{bmatrix}$$

11. (d) $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$\underset{\sim}{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_3 \to R_3(-1)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is in Echelon form.

The number of non-zero rows = 4

So, $\rho(A) = \text{rank of } A = 4$

12. (b) Augmented matrix of $Ax = b$

$$= A^* = \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 1 & 1 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to R_3 - R_1} \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & -1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 0 & 0 & -1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to R_3(-1)} \begin{bmatrix} 1 & 0 & 1 & 0 & \vdots & 0 \\ 0 & 1 & 0 & 1 & \vdots & 0 \\ 0 & 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

So, rank of $(A) = 4$ rank of A^* (augmented matrix)

$= 0(A)$ order of A

So, the system of equations is consistent having unique solution.

13. (b) Given, matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & 1 & -2 & 3 \\ 3 & 6 & 2 & -6 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 3 & -6 & 1 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \qquad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 5 & -12 & 2 \end{bmatrix} \qquad \left(R_2 \to \frac{1}{3}R_2\right)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 & 1 \\ 0 & 0 & 1 & -2 & 1/3 \\ 0 & 0 & 0 & -2 & 1/3 \end{bmatrix} \qquad (R_3 \to R_3 - 5R_2)$$

which is in Echelon form.

14. (d) From solution 13, $\rho(A) = 3$

15. (d) The given system of equations can be written in matrix form as $AX = B$

where, coefficient matrix

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix}$

$= 2(-3+2) - 3(3+6) - 1(1+3)$
$= -2 - 27 - 4$
$= -33 \neq 0$

$\therefore \rho(A) = 3$

16. (a) The system has only trivial solution.

17. (b) For any $n \times n$ matrix A with entries only 1, the characteristic polynomial is

$$\text{ch}_A(x) = x^{n-1}(x - nc)$$

Putting $n = 100$, we get

$$\text{ch}_A(x) = x^{100-1}(x - 100 \times 1)$$
$$= x^{99}(x - 100)$$

18. (d) Now, for given matrix A, we have

$$A^2 = 100A$$

$\Rightarrow \quad A^2 - 100A = 0$

$\Rightarrow$ Minimal polynomial

$$m_A(x) = x^2 - 100x$$

19. (b) Since, A is orthogonal.

$\Rightarrow \quad A'A = I$

$\Rightarrow \quad A(A'A) = A \cdot I = A$

$\Rightarrow \quad (AA')A = A$

$\Rightarrow \quad AA' = I$

Thus, $A'A = AA' = I$

$\Rightarrow$ A is non-singular.

$\Rightarrow$ A^{-1} exists and $A^{-1} = A'$

20. (c) Now, $\quad |A'A| = 1$

$\Rightarrow \quad |A'||A| = 1$

$\Rightarrow \quad |A|^2 = 1 \qquad (\because |A'| = |A|)$

$\Rightarrow \quad |A| = \pm 1$